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RESEARCH ARTICLE

Semi-Discretization of a Euler-Bernoulli Beam and Its Application to Motion Planning

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ABSTRACT We consider a Euler-Bernoulli beam with sliding cantilever boundary conditions at both ends. The control input to the beam is the force acting on one of the cantilevers. We derive an n^{th}-order semi-discrete approximation of the beam PDE and prove that the solution to the n^{th}-order semi-discrete system converges to the solution of the PDE as n tends to infinity. The motion planning problem addressed in this paper is to find a control input which will transfer the beam PDE from one steady state to another over a prescribed time interval. To address this problem, we design control inputs for transferring the semi-discrete systems from one steady state to another using the flatness technique. We show that a control input which solves the motion planning problem for the beam PDE can be obtained as a limit of a sequence of control inputs which solve certain motion planning problems for a sequence of semi-discrete systems of increasing order. We illustrate our theoretical results in simulations.

INDEX TERMS Euler-Bernoulli beam, flexible structure, finite-difference scheme, motion planning, semi-discretization.

I. INTRODUCTION

In the early lumping approach to the control of a dynamical system modelled by a partial differential equation (PDE), a set of ordinary differential equations (ODEs) in time called the semi-discrete approximation of the PDE is obtained by approximating the spatial derivatives in the PDE. Depending on the control objective, a finite-dimensional controller design technique is selected for constructing a control signal for the semi-discrete system. It is then shown that better is the approximation of the spatial derivatives, closer is the control signal constructed for the semi-discrete system to a limiting control signal which solves the control objective for the PDE. Therefore implementing the control signal constructed for a sufficiently accurate semi-discrete system on the dynamical system will result in satisfactory realization of the control objective.

An appealing feature of the early lumping approach is that existing finite-dimensional controller design techniques can be used to design control signals for the PDE. This approach has been used to design stabilizing controllers for PDEs by solving the algebraic Riccati equation [1], [2], [3], [4], [5], to design adaptive controllers for the heat equation using the backstepping technique [6], [7] and to study controllability and observability of Euler-Bernoulli beam PDEs [8], [9], [10], [11], and [12]. From these works it is evident that the properties of the numerical scheme used for obtaining the semi-discrete approximation of the PDE play an important role in the early lumping approach. The early lumping approach has been combined with the flatness technique to address motion planning problems for parabolic PDEs, see [13], [14], and [15]. Inspired by this, in the present paper we combine the early lumping approach with the flatness technique to address a motion planning problem for a Euler-Bernoulli beam PDE.

Consider the Euler-Bernoulli beam of unit length with sliding cantilever boundary conditions shown in Figure 1. The control input to the beam is the force acting on the cantilever located at the right end of the beam. The steady states of the beam are just the stationary undeformed (horizontal)

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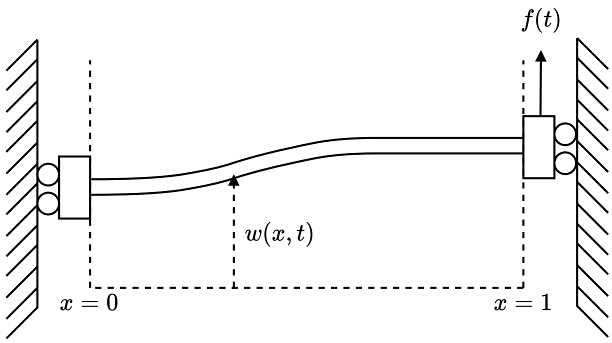

FIGURE 1. Schematic of a Euler-Bernoulli beam with sliding cantilever joints at both the ends.

configurations of the beam. In this paper, we address the motion planning problem of finding a control input which will transfer the beam from one steady state to another over a prescribed time interval. To solve this problem, we introduce a semi-discrete approximation of the beam PDE and establish its properties. Using the flatness technique, we solve the motion planning problem of finding a control input which will transfer the semi-discrete system from one steady state to another over a prescribed time interval. Finally, we show that a control input which solves the motion planning problem for the beam PDE can be obtained as the limit of a certain sequence of control inputs which solve the motion planning problem for a sequence of semi-discrete systems of increasing order. We remark that motion planning problems for Euler-Bernoulli beams with other boundary conditions have been solved by applying the flatness technique directly to the beam PDE without discretizing it, see for instance [16], [17], [18], [19], [20], and the same direct approach can potentially be used to solve the problem considered in this paper. However, our goal in this work is to explore whether and how early lumping approach can be used to solve this problem; This will provide insights for applying the early lumping approach to solve motion planning problems for more complex PDEs (higher-dimensional PDEs and nonlinear PDEs) when it is difficult to apply the flatness technique directly.

The main contributions of this paper are as follows: (i) We have derived an n^{th}-order semi-discrete approximation for the beam PDE using the finite-difference scheme and established its salient properties. Using these properties we have shown that the solution of the n^{th}-order semi-discrete system converges to the solution of the PDE as n tends to infinity. While we have proved this convergence for special initial states and inputs as needed in this work, the ideas in our proof can be adapted to establish the convergence for a larger class of initial states and inputs. (ii) We have shown that the motion planning problem considered in this work can be addressed via the early lumping approach. More specifically, we parameterize the inputs which solve the motion planning problem for the n^{th}-order semi-discrete system in terms of certain coefficients. We show that these coefficients converge to a limit as n tends to infinity and using these limits we find the input which solves the motion planning problem for the beam PDE. We remark that the semi-discrete approximation scheme that we have introduced can potentially form the basis for solving other control problems (different from motion planning) via the early lumping approach.

While proving that the solution to the n^{th}-order semi-discrete approximation of the PDE converges to the solution of the PDE (contribution (i) mentioned in the previous paragraph), we allow the flexural rigidity EI of the beam to be spatially varying, i.e. we consider a nonuniform beam. However, while proving that the control inputs derived for the semi-discrete systems converge to a control input which solves the motion planning problem for the PDE (contribution (ii) mentioned in the previous paragraph) we suppose that EI is constant, i.e. we consider a uniform beam. We remark that the control inputs derived for the semi-discrete systems converge to a control input which solves the motion planning problem for the PDE even when the beam is nonuniform, and we have demonstrated this numerically in this paper. We hope to establish this theoretically in a future work.

A preliminary and abridged version of some of the results presented in this paper have appeared (without proofs) in our conference paper [21] which considered Euler-Bernoulli beams with hinged boundary conditions. While the solution to the motion planning problem in [21] required two inputs, the solution in this paper requires only one input which is a significant improvement. Furthermore, this paper contains stronger claims supported by complete proofs.

The rest of the paper is organized as follows. Section II introduces the PDE model for the beam and the motion planning problem addressed in this work. In Section III we present our semi-discrete approximation for the nonuniform beam PDE. Our solutions to the motion planning problem for the semi-discrete system and the uniform beam PDE are presented in Sections IV and V, respectively. These sections also contain numerical illustrations of our theoretical results. In Section VI we demonstrate using a numerical example that our approach to solving the motion planning problem is also applicable to nonuniform beams. Finally, some concluding remarks are presented in Section VII. The notations used in this paper are introduced below.

Notations: Let $H^k(0,1)$ denote the usual Sobolev space of order $k \geq 1$ with the standard inner product. The space of continuous and k-times continuously differentiable functions from an interval $[a,b]$ to a Hilbert space X are denoted by $C([a,b];X)$ and $C^k([a,b];X)$, respectively, and they are both Banach spaces with the usual norm. The set of functions which belong to $C^k([a,b];X)$ for every $k \geq 0$ is denoted by $C^\infty([a,b];X)$. We write $C^k[a,b]$ instead of $C^k([a,b];\mathbb{R})$ and $C^\infty[a,b]$ instead of $C^\infty([a,b];\mathbb{R})$. The m^{th} derivative of a function $y \in C^\infty([a,b];X)$ is written as $y^{(m)}$. A function $y \in C^\infty[0,T]$ is said to be a Gevrey function of order $s > 0$ if it satisfies the estimate $\sup_{t\in[0,T]} |y^{(m)}(t)| \leq D^{m+1}(m!)^s$ for all $m \in \mathbb{N}$ and some constant $D > 0$. We denote the set of all functions satisfying these estimates by $G_s[0,T]$.

For any integer $n > 1$ and $v \in \mathbb{R}^n$, we denote the j^{th} component of v by $[v]_j$ and define $\|v\|_{2d} = \sqrt{hv^\top v}$

and $\|v\|_\infty = \max_{1\le i\le n} |[v]_i|$. Here $h = 1/(n-1)$. The discretization operator $R_n : C[0,1] \to \mathbb{R}^n$ is defined as follows: for any $z \in C[0,1]$, $R_n z = [z(0)\, z(h) \,\cdots\, z(nh-h)]^\top$.

II. BEAM MODEL, SOLUTION AND PROBLEM STATEMENT

Consider the following model for a Euler-Bernoulli beam of unit length which has sliding cantilever boundary conditions at both the ends: For $t \ge 0$,

$$w_{tt}(x,t) + (EIw_{xx})_{xx}(x,t) = 0 \qquad \forall\, x \in (0,1), \quad (1)$$

$$w_x(0,t) = 0, \quad (EIw_{xx})_x(0,t) = 0, \quad (2)$$

$$w_x(1,t) = 0, \quad (EIw_{xx})_x(1,t) = f(t). \quad (3)$$

Here $w(x,t)$ is the displacement of the beam at the location $x \in [0,1]$ and time $t \ge 0$, the strictly positive function $EI \in C^4[0,1]$ is the flexural rigidity of the beam and $f(t)$ is the input force acting on the cantilever joint at $x = 1$. A schematic depicting the beam and the boundary conditions is shown in Figure 1. Let $T > 0$. A function $w \in C^2([0,T]; L^2(0,1)) \cap C([0,T]; H^4(0,1))$ is said to be a *solution* of the above beam model if it satisfies (1) for each $t \in [0,T]$ and almost every $x \in (0,1)$ and satisfies (2)-(3) for each $t \in [0,T]$ and some input $f \in C[0,T]$. We introduce the semigroup associated with (1)-(3) in the next paragraph and then present an expression for the solutions of (1)-(3) using this semigroup. The motion planning problem addressed in this paper is introduced at the end of this section.

The differential operator associated with the PDE (1)-(3) is $\mathcal{P} : L^2(0,1) \mapsto L^2(0,1)$ whose domain is

$$\mathcal{D}(\mathcal{P}) = \Big\{u \in H^4(0,1) \,\Big|\, u_x(0) = (EIu_{xx})_x(0) = 0, \\ u_x(1) = (EIu_{xx})_x(1) = 0\Big\}$$

and $\mathcal{P}w = (EIw_{xx})_{xx}$ for all $w \in \mathcal{D}(\mathcal{P})$. It follows via integration by parts that the operator $\mathcal{P}$ is self adjoint and non-negative. So $\mathcal{P} + I$ (where I is the identity operator on $L^2(0,1)$) is self-adjoint and positive (coercive) and $\mathcal{D}(\mathcal{P}) = \mathcal{D}(\mathcal{P}+I)$. Let $(\mathcal{P}+I)^{\frac{1}{2}}$ be the unique positive square root of $\mathcal{P}+I$. Consider the space $Z = \mathcal{D}((\mathcal{P}+I)^{\frac{1}{2}}) \times L^2(0,1)$ with the inner product

$$\langle z_1, z_2\rangle_Z = \langle (\mathcal{P}+I)^{\frac{1}{2}} u_1, (\mathcal{P}+I)^{\frac{1}{2}} u_2\rangle_{L^2} + \langle v_1, v_2\rangle_{L^2} \quad (4)$$

for all $z_1 = [u_1 \;\; v_1]^\top \in Z$ and $z_2 = [u_2 \;\; v_2]^\top \in Z$. The norm on Z induced by this inner product is

$$\|z\|_Z^2 = \|(\mathcal{P}+I)^{\frac{1}{2}} u\|_{L^2}^2 + \|v\|_{L^2}^2 \qquad \forall\, z = \begin{bmatrix} u \\ v \end{bmatrix} \in Z. \quad (5)$$

Then Z is a Hilbert space and the operator

$$\begin{bmatrix} 0 & I \\ -\mathcal{P} - I & 0 \end{bmatrix} \quad (6)$$

with domain $\mathcal{D}(\mathcal{P}) \times \mathcal{D}((\mathcal{P}+I)^{\frac{1}{2}}) \subset Z$ is the generator of a strongly continuous semigroup on Z, see [22, Example 2.2.5]. Hence the operator

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -\mathcal{P} & 0 \end{bmatrix},$$

which differs from the operator in (6) by a bounded perturbation, also has domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{P}) \times \mathcal{D}((\mathcal{P}+I)^{\frac{1}{2}}) \subset Z$ and is the generator of a strongly continuous semigroup $\mathbb{T}$ on Z. Below we present an expression for the solutions of (1)-(3) in terms of $\mathbb{T}$, see (15).

Let $\mu \in C^\infty[0,1]$ be such that

$$\mu_x(0) = (EI\mu_{xx})_x(0) = \mu_x(1) = 0, \quad (EI\mu_{xx})_x(1) = 1. \quad (7)$$

Fix $T > 0$ and input $f \in C^\infty[0,T]$ such that

$$f(0) = \dot{f}(0) = 0. \quad (8)$$

Define

$$\tilde{w}(x,t) = w(x,t) - \mu(x) f(t) \qquad \forall\, x \in [0,1],\; t \in [0,T]. \quad (9)$$

Substituting $w = \tilde{w} + \mu f$ in (1)-(3) we get that $\tilde{w}$ satisfies the following PDE: For $t \in [0,T]$,

$$\tilde{w}_{tt}(x,t) + (EI\tilde{w}_{xx})_{xx}(x,t) + (EI\mu_{xx})_{xx}(x) f(t) \\ + \mu(x)\ddot{f}(t) = 0 \qquad \forall x \in (0,1), \quad (10)$$

$$\tilde{w}_x(0,t) = 0, \qquad (EI\tilde{w}_{xx})_x(0,t) = 0, \quad (11)$$

$$\tilde{w}_x(1,t) = 0, \qquad (EI\tilde{w}_{xx})_x(1,t) = 0. \quad (12)$$

Denote $w(\cdot,t)$ by $w(t)$ and $\tilde{w}(\cdot,t)$ by $\tilde{w}(t)$. The above PDE can be rewritten as an abstract evolution equation on the state space Z as follows: For $t \in [0,T]$,

$$\begin{bmatrix} \tilde{w}_t(t) \\ \tilde{w}_{tt}(t) \end{bmatrix} = \mathcal{A} \begin{bmatrix} \tilde{w}(t) \\ \tilde{w}_t(t) \end{bmatrix} + \mathcal{F}(t), \quad (13)$$

where $\mathcal{F} = [0 \;\; -(EI\mu_{xx})_{xx} f - \mu\ddot{f}]^\top \in C^\infty([0,T]; Z)$. Given an initial state $[u \;\; v]^\top \in \mathcal{D}(\mathcal{A})$, we can apply the regularity result [22, Theorem 3.1.3] to (13) and conclude that $\tilde{w}$ given by the variation of constants formula

$$\begin{bmatrix} \tilde{w}(t) \\ \tilde{w}_t(t) \end{bmatrix} = \mathbb{T}_t \begin{bmatrix} u \\ v \end{bmatrix} + \int_0^t \mathbb{T}_\tau \mathcal{F}(t-\tau)\, d\tau \qquad \forall\, t \in [0,T] \quad (14)$$

belongs to $C^2([0,T]; L^2(0,1)) \cap C([0,T]; H^4(0,1))$, it satisfies (10) for each $t \in [0,T]$ and almost every $x \in (0,1)$, it satisfies (11)-(12) for each $t \in [0,T]$, and $\tilde{w}(0) = u$ and $\tilde{w}_t(0) = v$. Using this, the relationship between w and $\tilde{w}$ in (9) and the properties of μ and f in (7)-(8) we can conclude via a simple calculation that w defined as

$$\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = \mathbb{T}_t \begin{bmatrix} u \\ v \end{bmatrix} + \int_0^t \mathbb{T}_\tau \mathcal{F}(t-\tau)\, d\tau + \begin{bmatrix} \mu f(t) \\ \mu \dot{f}(t) \end{bmatrix} \quad (15)$$

for all $t \in [0,T]$ belongs to $C^2([0,T]; L^2(0,1)) \cap C([0,T]; H^4(0,1))$, it satisfies (1) for each $t \in [0,T]$ and almost every $x \in (0,1)$, it satisfies (2)-(3) for each $t \in [0,T]$, and $w(0) = u$ and $w_t(0) = v$. In other words, w given by (15) is the solution of (1)-(3) corresponding to the initial state $[u \;\; v]^\top$ and input f. In the next proposition we will show under certain additional hypothesis on the initial state and input that $w \in C^\infty([0,T]; C^5[0,1])$.

Proposition 1. *Fix $T > 0$. Suppose that the initial state $[u \;\; v]^\top \in \mathcal{D}(\mathcal{A}^\infty)$ and the input $f \in C^\infty[0,T]$ with*

$$f^{(k)}(0) = 0 \qquad \forall\, k \geq 0. \tag{16}$$

Then the solution w of the PDE (1)-(3), corresponding to this initial state and input, given by (15) belongs to $C^\infty([0,T]; C^5[0,1])$.

Proof: Using (16) in the definition of $\mathcal{F}$ given below (13) we get that $\mathcal{F} \in C^\infty([0,T]; Z)$ and

$$\mathcal{F}^{(k)}(0) = 0 \qquad \forall\, k \geq 0. \tag{17}$$

Differentiating (15) k-times with respect to time and then using (17) and the definition of $\mathcal{F}$ we get

$$\begin{bmatrix} w^{(k)}(t) \\ w^{(k+1)}(t) \end{bmatrix} = \mathbb{T}_t \mathcal{A}^k \begin{bmatrix} u \\ v \end{bmatrix} + \int_0^t \mathbb{T}_\tau \mathcal{F}^{(k)}(t-\tau)d\tau + \begin{bmatrix} \mu f^{(k)}(t) \\ \mu f^{(k+1)}(t) \end{bmatrix} \tag{18}$$

for all $t \in [0,T]$ and $k \geq 0$. The three terms on the right side of the above expression resemble the three terms on the right side of (15) (with $\mathcal{A}^k[u \;\; v]^\top \in \mathcal{D}(\mathcal{A})$ instead of $[u \;\; v]^\top \in \mathcal{D}(\mathcal{A})$, $\mathcal{F}^{(k)} \in C^\infty([0,T]; Z)$ instead of $\mathcal{F} \in C^\infty([0,T]; Z)$ and $f^{(k)} \in C^\infty[0,T]$ instead of $f \in C^\infty[0,T]$). Therefore via the reasoning that enabled us to conclude that w given in (15) belongs to $C([0,T]; H^4(0,1))$, we get that $w^{(k)}$ given in (18) belongs to $C([0,T]; H^4(0,1))$. Since this is true for all integers $k \geq 0$ we have

$$w \in C^\infty([0,T]; H^4[0,1]). \tag{19}$$

Now since w satisfies (1), i.e. $(EIw_{xx})_{xx}(t) = -w_{tt}(t)$ for each $t \in [0,T]$, we can conclude using (19) that $EIw_{xx} \in C^\infty([0,T]; H^6[0,1])$ which together with $EI \in C^4[0,1]$ implies that $w \in C^\infty([0,T]; C^5[0,1])$. This completes the proof. □

A *steady state* of the PDE (1)-(3) is a vector $[u \;\; v]^\top \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Using the definition of $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$ it is easy to see that all the steady states of (1)-(3) are of the form $[u \;\; 0]^\top$, where u is a constant function on the interval $[0,1]$. Suppose that $[w_0 \;\; 0]^\top$ is a steady state of $\mathcal{A}$, then

$$\mathbb{T}_t \begin{bmatrix} w_0 \\ 0 \end{bmatrix} = \begin{bmatrix} w_0 \\ 0 \end{bmatrix} \qquad \forall\, t \geq 0. \tag{20}$$

We say that f transfers (1)-(3) from $[u_0 \;\; v_0]^\top$ to $[u_T \;\; v_T]^\top$ over the time interval $[0,T]$ if the solution w of (1)-(3) for the initial state $[u_0 \;\; v_0]^\top$ and input f satisfies $[w(T) \;\; w_t(T)]^\top = [u_T \;\; v_T]^\top$. In this paper, given $T > 0$ and steady states $[w_0 \;\; 0]^\top$ and $[w_T \;\; 0]^\top$, we are interested in the problem of finding an input $f \in C^\infty[0,T]$ which transfers (1)-(3) from $[w_0 \;\; 0]^\top$ to $[w_T \;\; 0]^\top$ over the time interval $[0,T]$. However, we will assume that the initial state is zero and then address this problem. This does not lead to any loss of generality since if f transfers (1)-(3) from the zero state to $[w_T - w_0 \;\; 0]^\top$ over the time interval $[0,T]$, then using (20) and (15) it is easy to verify that the same f transfers (1)-(3) from $[w_0 \;\; 0]^\top$ to $[w_T \;\; 0]^\top$ over the time interval $[0,T]$. To summarize we address the following problem in this paper:

Problem 1. *Given $T > 0$ and a steady state $[w_T \;\; 0]^\top$ of the PDE (1)-(3), find an input $f \in C^\infty[0,T]$ which transfers the PDE from the zero state to $[w_T \;\; 0]^\top$ over the time interval $[0,T]$.*

In Section V we prove that there is a solution to the above problem for any $T > 0$ and any steady state $[w_T \;\; 0]^\top$ when EI is a constant. Using a numerical example we demonstrate in Section VI that our solution is also applicable when EI is spatially varying.

Remark 1. *Suppose that EI is a constant in the Euler-Bernoulli beam PDE with sliding cantilever boundary conditions (1)-(3). Then the change of variable $u = w_x$ transforms (1)-(3) to a beam PDE with hinged boundary conditions, and motion planning for such beams has been considered in [9]. Under this transformation all the steady states of the sliding cantilever beam are mapped to the zero function, and so Problem 1 cannot be addressed by reformulating it as a motion planning problem for the transformed hinged beam.*

III. FINITE-DIFFERENCE SEMI-DISCRETIZATION

In this section we derive a semi-discrete approximation for the PDE (1)-(3) by replacing the spatial derivatives in (1)-(3) with their finite-difference approximations. We then show that the solution of the semi-discrete approximation converges to the solution of the PDE as the discretization step-size converges to zero.

Consider a function $w \in C^2([0,T]; C^5[0,1])$ which satisfies the PDE (1)-(3). Let $h = 1/(n-1)$, where $n \geq 5$ is an integer. Then for each $j \in \{2, 3, \ldots n-3\}$ we get using Taylor's theorem that

$$\begin{aligned}
&(EIw_{xx})_{xx}(jh,t) \\
&= EI(jh-h)\frac{w(jh-2h,t) - 2w(jh-h,t) + w(jh,t)}{h^4} \\
&\quad - 2EI(jh)\frac{w(jh-h,t) - 2w(jh,t) + w(jh+h,t)}{h^4} \\
&\quad + EI(jh+h)\frac{w(jh) - 2w(jh+h) + w(jh+2h)}{h^4} + \mathcal{O}(jh,t),
\end{aligned}$$

where $\mathcal{O}(jh,t)$ satisfies the bound

$$\sup_{t\in[0,T]} |\mathcal{O}(jh,t)| \leq Kh \sup_{t\in[0,T]} \|w(\cdot,t)\|_{C^5[0,1]} \tag{21}$$

for some $K > 0$ independent of h, j and w. Using Taylor's theorem again, along with the boundary conditions (2)-(3), we get the following expressions for $(EIw_{xx})_{xx}(jh,t)$ for $j \in \{0, 1, n-2, n-1\}$:

$$(EIw_{xx})_{xx}(0,t) = EI(h)\frac{w(0,t) - 2\,w(h,t) + w(2h,t)}{h^4}$$

$$- \left(EI(0) + \frac{hEI_x(0)}{3} \right) \frac{2w(h,t) - 2w(0,t)}{h^4} + \mathcal{O}(0,t),$$

$$(EIw_{xx})_{xx}(h,t) = EI(2h)\frac{w(h,t) - 2\,w(2h,t) + w(3h,t)}{h^4} - 2EI(h)\frac{w(0,t) - 2\,w(h,t) + w(2h,t)}{h^4} + \left(EI(0) + \frac{hEI_x(0)}{3} \right) \frac{2w(h,t) - 2w(0,t)}{h^4} + \mathcal{O}(h,t).$$

$$(EIw_{xx})_{xx}(nh-2h,t) = EI(1-2h)\frac{w(nh-4h,t) - 2\,w(nh-3h,t) + w(nh-2h,t)}{h^4} - 2EI(1-h)\frac{w(nh-3h,t) - 2\,w(nh-2h,t) + w(nh-h,t)}{h^4} + \left(EI(1) - \frac{hEI_x(1)}{3} \right) \frac{2w(nh-2h,t) - 2w(nh-h,t)}{h^4} + \frac{f(t)}{3h} + \mathcal{O}(nh-h,t).$$

$$(EIw_{xx})_{xx}(nh-h,t) = EI(1-h)\frac{w(nh-h,t) - 2\,w(nh-2h,t) + w(nh-3h,t)}{h^4} - \left(EI(1) - \frac{hEI_x(1)}{3} \right) \frac{2w(nh-2h,t) - 2w(nh-h,t)}{h^4} + \frac{2f(t)}{3h} + \mathcal{O}(nh-2h,t).$$

Here $\mathcal{O}(jh,t)$ satisfies the following bound:

$$\sup_{t\in[0,T]} |\mathcal{O}(jh,t)| \le K \sup_{t\in[0,T]} \|w(\cdot,t)\|_{C^5[0,1]} \tag{22}$$

for $j \in \{0, 1, n-2, n-1\}$ and some $K > 0$ independent of h, j and w. Let

$$v_n = \left[w(0,t)\ w(h,t)\ \cdots\ w(nh-h,t) \right]^\top .$$

From (1) it follows that

$$\ddot{v}_n(t) = -\left[(EIw_{xx})_{xx}(0,t)\ \cdots\ (EIw_{xx})_{xx}(nh-h,t) \right]^\top .$$

Substituting the expressions derived earlier for $(EIw_{xx})_{xx}(jh,t)$ in to the above equation we get

$$\ddot{v}_n(t) = -P_n v_n(t) + B_n f(t) + \mathcal{O}_n(t), \tag{23}$$

where the j^{th} entry of $\mathcal{O}_n(t) \in \mathbb{R}^n$ is $-\mathcal{O}(jh-h,t)$ and the matrices $P_n \in \mathbb{R}^{n\times n}$ and $B_n \in \mathbb{R}^n$ are as given below: $P_n = L_n^\top E_n L_n$ where

$$L_n = \frac{1}{h^2} \begin{bmatrix} -\sqrt{2} & \sqrt{2} & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}, \tag{24}$$

$$E_n = \operatorname{diag}\left[EI(0) + \tfrac{hEI_x(0)}{3}\ EI(h)\ \cdots\ EI(1-h)\ EI(1) - \tfrac{hEI_x(1)}{3} \right],$$

$$B_n = \frac{1}{h}\left[0\ 0\ \cdots\ -\tfrac{1}{3}\ -\tfrac{2}{3} \right]^\top .$$

Dropping the correction term $\mathcal{O}_n$ from (23) and then writing it as a first-order equation we obtain the following semi-discrete approximation for (1)-(3) when the input $f = f_n$:

$$\begin{bmatrix} \dot{v}_n(t) \\ \ddot{v}_n(t) \end{bmatrix} = A_n \begin{bmatrix} v_n(t) \\ \dot{v}_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_n \end{bmatrix} f_n(t) \qquad \forall\, t \ge 0, \tag{25}$$

where

$$A_n = \begin{bmatrix} 0 & I \\ -P_n & 0 \end{bmatrix}. \tag{26}$$

Observe that the first and the last entries of the matrix E_n defined below (24) are larger than a strictly positive constant independent of n for all n sufficiently large. We will assume that this is true for all n to simplify our presentation. Dropping this assumption will not affect any of our results since we are only concerned with the solutions of (25) as $n \to \infty$.

The following lemma summarizes the accuracy of the Taylor's theorem based finite-difference approximations presented above in a form that is subsequently used in this paper. Recall $h = 1/(n-1)$, the discretization operator R_n, and the norm $\|\cdot\|_{2d}$ from the notations in Section I.

Lemma 1. *Fix $T > 0$. Suppose that $\xi \in C([0,T]; C^5[0,1])$ satisfies $\xi_x(0,t) = 0$, $(EI\xi_{xx})_x(0,t) = 0$ and $\xi_x(1,t) = 0$. Define $f_\xi(t) = (EI\xi_{xx})_x(1,t)$. Then there exists a constant $c > 0$ independent of n and ξ such that*

$$\sup_{t\in[0,T]} \|R_n(EI\xi_{xx})_{xx}(\cdot,t) - P_n R_n \xi(\cdot,t) + B_n f_\xi(t)\|_{2d} \le c\sqrt{h} \sup_{t\in[0,T]} \|\xi(\cdot,t)\|_{C^5[0,1]} \tag{27}$$

for all $n \ge 5$.

Proof: For a vector $v \in \mathbb{R}^n$, recall that $[v]_j$ denote its j^{th} component. From the expressions above (21) and (22) (with ξ in place of w) and the definitions of the matrices P_n and B_n we get

$$[R_n(EI\xi_{xx})_{xx}(\cdot,t) - P_n R_n \xi(\cdot,t) + B_n f_\xi(t)]_{j+1} = \mathcal{O}(jh,t) \tag{28}$$

for $j \in \{0, 1, \ldots n-1\}$ and from (21) and (22) we get that there exists a constant $K > 0$ independent of h, j and ξ such that

$$\sup_{t\in[0,T]} |\mathcal{O}(jh,t)| \le Kh \sup_{t\in[0,T]} \|\xi(\cdot,t)\|_{C^5[0,1]} \tag{29}$$

for $j \in \{2, 3, \ldots n-3\}$ and

$$\sup_{t\in[0,T]} |\mathcal{O}(jh,t)| \le K \sup_{t\in[0,T]} \|\xi(\cdot,t)\|_{C^5[0,1]} \tag{30}$$

for $j \in \{0, 1, n-2, n-1\}$. The estimate in (27) now follows immediately from (28)-(30) and the definition of $\|\cdot\|_{2d}$. ☐

The next lemma presents a type of discrete Sobolev inequality needed in the proof of our convergence result Theorem 2.

Lemma 2. *For each $\epsilon > 0$, there exists an $N(\epsilon) > 0$ independent of n such that*

$$\|v\|_\infty^2 \leq \epsilon \|L_n v\|_{2d}^2 + N(\epsilon)\|v\|_{2d}^2 \qquad \forall\, v \in \mathbb{R}^n. \tag{31}$$

Proof: From [13, Eq. (3.19)] we get that for each $\epsilon > 0$, there exists a $C(\epsilon) > 0$ independent of n such that

$$\|v\|_\infty^2 \leq \epsilon \sum_{i=1}^{n-1} \frac{([v]_{i+1} - [v]_i)^2}{h} + C(\epsilon)\|v\|_{2d}^2 \qquad \forall\, v \in \mathbb{R}^n. \tag{32}$$

Recall that $[v]_i$ denotes the i^{th}-component of the vector v. (Note that [13, Eq. (3.19)] is in terms of $\|\cdot\|_2^2$ while the above equation is in terms of $\|\cdot\|_{2d}^2$ and $\|\cdot\|_{2d}^2 = h\|\cdot\|_2^2$.)

Consider the diagonal matrix

$$\Gamma_n = \operatorname{diag}\left[\tfrac{1}{\sqrt{2}}\ 1 \cdots 1\ \tfrac{1}{\sqrt{2}}\right] \in \mathbb{R}^{n\times n}.$$

A simple calculation using the definition of L_n gives

$$\sum_{i=1}^{n-1} \frac{([v]_{i+1} - [v]_i)^2}{h} = -hv^\top \Gamma_n L_n v$$

Bounding the term $-hv^\top \Gamma_n L_n v$ on the right-side of the above equation using the Cauchy-Schwartz inequality and the Young's inequality we get

$$\sum_{i=1}^{n-1} \frac{([v]_{i+1} - [v]_i)^2}{h} \leq \|L_n v\|_{2d}^2 + \|v\|_{2d}^2.$$

The estimate (31) now follows from (32) and the above inequality. □

For the semi-discrete system (25), we take the state space to be $Z_d = \mathbb{R}^n \times \mathbb{R}^n$ with the following inner product:

$$\left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle_{Z_d} = hu_1^\top v_1 + hu_1^\top L_n^\top E_n L_n v_1 + hu_2^\top v_2$$

for all $[u_1\ u_2]^\top \in Z_d$ and $[v_1\ v_2]^\top \in Z_d$. So the norm of $[u_1\ u_2]^\top \in Z_d$ is

$$\left\| \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|_{Z_d}^2 = \|u_1\|_{2d}^2 + \|E_n^{1/2} L_n u_1\|_{2d}^2 + \|u_2\|_{2d}^2. \tag{33}$$

Recall A_n from (26) for $n \geq 5$, and $P_n = L_n^\top E_n L_n$ from above (24). It is easy to see that

$$\left\langle A_n \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\rangle_{Z_d} = hu_2^\top u_1 \leq \frac{1}{2} \left\| \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|_{Z_d}^2$$

for all $[u_1\ u_2]^\top \in Z_d$. Let A_n^* be the adjoint of A_n. Then for each $z \in Z_d$ we have $\langle A_n^* z, z\rangle_{Z_d} = \langle z, A_n z\rangle_{Z_d}$ by definition and $\langle z, A_n z\rangle_{Z_d} = \langle A_n z, z\rangle_{Z_d}$ since the inner product is symmetric. So $\langle A_n^* z, z\rangle_{Z_d} = \langle A_n z, z\rangle_{Z_d}$. Using this and the above inequality, it follows from [22, Corollary 2.2.3] that

$$\|e^{A_n t}\| \leq e^{\frac{t}{2}} \qquad \forall\, t \geq 0, \quad \forall\, n \geq 5. \tag{34}$$

In the above estimate, $\|e^{A_n t}\|$ is the matrix norm of $e^{A_n t}$ induced by the norm on Z_d.

The following theorem is the main result of this section. It shows that when the initial state of the PDE (1)-(3) is zero and the input satisfies certain smoothness assumptions, the solution of the n^{th}-order semi-discrete approximation (25) converges pointwise to the solution of the PDE (1)-(3) as $n \to \infty$. We remark that the ideas in the proof of this theorem can be adapted to establish convergence results under milder hypothesis on the initial state and input. Recall the discretization operator R_n from the notations in Section I.

Theorem 2. *Fix $T > 0$. Consider a function $f \in C^\infty[0,T]$ and a sequence of functions $\{f_n\}_{n\geq 0}$ in $C^\infty[0,T]$ satisfying $f^{(k)}(0) = f_n^{(k)}(0) = 0$ for all $k \geq 0$ and*

$$\lim_{n\to\infty} \|f_n - f\|_{C^4[0,T]} = 0. \tag{35}$$

Let w be the solution of (1)-(3) on the interval $[0,T]$ with the initial state $w(\cdot,0) = 0$, $w_t(\cdot,0) = 0$ and input f. For $n \geq 5$, let $\left[\begin{smallmatrix} v_n \\ \dot v_n \end{smallmatrix}\right]$ be the solution of the n^{th}-order semi-discrete system (25) on the interval $[0,T]$ with zero initial state and input f_n. Then

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| \begin{bmatrix} R_n w(\cdot,t) \\ R_n w_t(\cdot,t) \end{bmatrix} - \begin{bmatrix} v_n(t) \\ \dot v_n(t) \end{bmatrix} \right\|_\infty = 0. \tag{36}$$

Proof: From Proposition 1 we get that the solution w of the PDE (1)-(3) corresponding to zero initial state and input f belongs to $C^\infty([0,T]; C^5[0,1])$ and it satisfies (1)-(3) pointwise.

Let $\mu \in C^\infty[0,1]$ be the function defined above (7). Define the error to be

$$e_n(t) = R_n w(\cdot,t) - v_n(t) - R_n\mu[f(t) - f_n(t)].$$

For $t \in [0,T]$, using (1) and (25), we can compute that

$$\begin{bmatrix} \dot e_n(t) \\ \ddot e_n(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -P_n & 0 \end{bmatrix} \begin{bmatrix} e_n(t) \\ \dot e_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \delta_n(t) \end{bmatrix}, \tag{37}$$

$$\begin{bmatrix} e_n(0) \\ \dot e_n(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{38}$$

where $\delta_n(t) = \delta_n^1(t) + \delta_n^2(t) + \delta_n^3(t)$ with

$$\delta_n^1(t) = P_n R_n w(\cdot,t) - R_n(EIw_{xx})_{xx}(\cdot,t) - B_n f(t),$$
$$\delta_n^2(t) = (P_n R_n \mu - B_n)[f_n(t) - f(t)],$$
$$\delta_n^3(t) = [\ddot f_n(t) - \ddot f(t)]R_n\mu.$$

The solution of (37)-(38) is

$$\begin{bmatrix} e_n(t) \\ \dot e_n(t) \end{bmatrix} = \int_0^t e^{A_n\tau} \begin{bmatrix} 0 \\ \delta_n(t-\tau) \end{bmatrix} d\tau. \tag{39}$$

Since $w(\cdot,0) = 0$, $f(0) = \ddot f(0) = 0$ and $f_n(0) = \ddot f_n(0) = 0$ (see the theorem statement), it follows from the definition of δ_n that $\delta_n(0) = 0$. Differentiating the above equation and using $\delta_n(0) = 0$ we get

$$\begin{bmatrix} \dot e_n(t) \\ \ddot e_n(t) \end{bmatrix} = \int_0^t e^{A_n\tau} \begin{bmatrix} 0 \\ \dot\delta_n(t-\tau) \end{bmatrix} d\tau. \tag{40}$$

Using the estimate in (34) to bound the right-side of (39) and (40) we get that

$$\sup_{t\in[0,T]}\left(\left\|\begin{bmatrix}e_n(t)\\ \dot e_n(t)\end{bmatrix}\right\|_{Z_d}+\left\|\begin{bmatrix}\dot e_n(t)\\ \ddot e_n(t)\end{bmatrix}\right\|_{Z_d}\right)\leq K\sup_{t\in[0,T]}\left(\left\|\begin{bmatrix}0\\ \delta_n(t)\end{bmatrix}\right\|_{Z_d}+\left\|\begin{bmatrix}0\\ \dot\delta_n(t)\end{bmatrix}\right\|_{Z_d}\right) \tag{41}$$

for some constant K independent of n. Next we derive certain estimates for δ_n and $\dot\delta_n$ which can be used along with (41) to complete the proof of this theorem.

Recall δ_n, δ_n^1, δ_n^2 and δ_n^3 introduced below (38). Since w satisfies all the hypothesis imposed on ξ in Lemma 1 with $(EIw_{xx})_x(1,t)=f(t)$ (see the beginning of this proof), we can conclude by applying the lemma to w that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\delta_n^1(t)\|_{2d}=0. \tag{42}$$

Differentiating the expression for δ_n^1 we get $\dot\delta_n^1(t) = P_nR_nw_t(\cdot,t)-R_n(EIw_{txx})_{xx}(\cdot,t)-B_n\dot f(t)$. The regularity of w implies that w_t also satisfies all the hypothesis imposed on ξ in Lemma 1 with $(EIw_{txx})_x(1,t)=\dot f(t)$. So we can conclude by applying the lemma to w_t that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\dot\delta_n^1(t)\|_{2d}=0. \tag{43}$$

Let us denote the constant (in time) function belonging to $C^\infty([0,T];C^5[0,1])$ whose value at each time instant $t\in[0,T]$ is μ, also by μ. Then μ satisfies all the hypothesis imposed on ξ in Lemma 1 with $(EI\mu_{xx})_x(1,t)=1$ and we can conclude by applying the lemma to μ that $\lim_{n\to\infty}\|P_nR_n\mu-R_n(EI\mu_{xx})_{xx}-B_n\|_{2d}=0$. From this and the fact that $\|R_n(EI\mu_{xx})_{xx}\|_{2d}$ can be bounded by a constant which depends only on $\max_{x\in[0,1]}|(EI\mu_{xx})_{xx}(x)|$, it follows that $\|P_nR_n\mu-B_n\|_{2d}<C$ for some $C>0$ independent of n. Using this and the limit (35) gives

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\delta_n^2(t)\|_{2d}=\lim_{n\to\infty}\sup_{t\in[0,T]}\|\dot\delta_n^2(t)\|_{2d}=0. \tag{44}$$

Finally from (35) and the fact that $\|R_n\mu\|_{2d}$ can be bounded by a constant which depends only on $\max_{x\in[0,1]}|\mu(x)|$, we get

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\delta_n^3(t)\|_{2d}=\lim_{n\to\infty}\sup_{t\in[0,T]}\|\dot\delta_n^3(t)\|_{2d}=0. \tag{45}$$

Combining the limits in (42)-(45) and recalling the definition of $\|\cdot\|_{Z_d}$ from (33) we obtain

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|\begin{bmatrix}0\\ \delta_n(t)\end{bmatrix}\right\|_{Z_d}=\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|\begin{bmatrix}0\\ \dot\delta_n(t)\end{bmatrix}\right\|_{Z_d}=0. \tag{46}$$

We will now complete the proof of this theorem. From the estimate in (41) and the limits in (46) it is easy to conclude that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|\begin{bmatrix}e_n(t)\\ \dot e_n(t)\end{bmatrix}\right\|_{Z_d}=\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|\begin{bmatrix}\dot e_n(t)\\ \ddot e_n(t)\end{bmatrix}\right\|_{Z_d}=0.$$

The above limits together with the definition of $\|\cdot\|_{Z_d}$ given in (33) imply that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|e_n(t)\|_{2d}=\lim_{n\to\infty}\sup_{t\in[0,T]}\|E_n^{1/2}L_ne_n(t)\|_{2d}=0,$$
$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\dot e_n(t)\|_{2d}=\lim_{n\to\infty}\sup_{t\in[0,T]}\|E_n^{1/2}L_n\dot e_n(t)\|_{2d}=0,$$

Since E_n is a diagonal matrix whose entries are larger than a strictly positive constant independent of n we can rewrite the above limits as follows:

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|e_n(t)\|_{2d}=\lim_{n\to\infty}\sup_{t\in[0,T]}\|L_ne_n(t)\|_{2d}=0,$$
$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\dot e_n(t)\|_{2d}=\lim_{n\to\infty}\sup_{t\in[0,T]}\|L_n\dot e_n(t)\|_{2d}=0,$$

The above limits then in turn imply, via Lemma 2, that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|e_n(t)\|_\infty=\lim_{n\to\infty}\sup_{t\in[0,T]}\|\dot e_n(t)\|_\infty=0.$$

The limit in (36) now follows from the above limits, the definitions of $e_n(t)$ and $\dot e_n(t)$ and the limit in (35). □

IV. MOTION PLANNING FOR THE SEMI-DISCRETE APPROXIMATION WITH CONSTANT *EI*

In this section we assume that the flexural rigidity EI of the beam is a constant. Without loss of generality we take $EI=1$ so that E_n is the $n\times n$ identity matrix.

A steady-state of the finite-dimensional semi-discrete system (25) is any vector $v_{ss}\in\mathbb{R}^{2n}$ which satisfies $A_nv_{ss}=0$. Suppose that $v_{ss}=\left[\begin{smallmatrix}v_{1,ss}\\ v_{2,ss}\end{smallmatrix}\right]$ with $v_{1,ss},v_{2,ss}\in\mathbb{R}^n$. Then from the definition of A_n in (26) it follows that

$$A_nv_{ss}=0\quad\iff\quad P_nv_{1,ss}=0,\quad v_{2,ss}=0.$$

Recall that $P_n=L_n^\top E_nL_n=L_n^\top L_n$, where L_n is defined in (24). Therefore $P_nv_{1,ss}=0$ implies that $L_n^\top L_nv_{1,ss}=0$, which in turn implies that $v_{1,ss}^\top L_n^\top L_nv_{1,ss}=0$ or equivalently that $L_nv_{1,ss}=0$. From the definition of L_n, it is easy to see that $L_nv_{1,ss}=0$ if and only if

$$v_{1,ss}=\begin{bmatrix}\alpha & \alpha & \cdots & \alpha\end{bmatrix}^\top \quad\text{for some }\alpha\in\mathbb{R}. \tag{47}$$

Therefore the steady states of (25) are all vectors of the form $\left[\begin{smallmatrix}v_{1,ss}\\ 0\end{smallmatrix}\right]$, where $v_{1,ss}\in\mathbb{R}^n$ is as given in (47). In this section, we present an algorithm for finding an input f_n which transfers the finite-dimensional semi-discrete system (25) from the zero state to any other steady state over the time interval $[0,T]$ for some $T>0$. Throughout this section, we suppose that $n\geq 5$.

We will now rewrite the semi-discrete system (25) in a form useful for finding an input f_n which transfers (25) between the given steady states. Observe that (25) can equivalently be written as

$$\ddot v_n(t)=-P_nv_n(t)+B_nf_n(t)\qquad\forall\, t\geq 0, \tag{48}$$

which is a collection of n scalar differential equations. By rearranging the terms in each of these equations we obtain the following set of n equations which are equivalent to (25):

$$[v_n]_3=-h^4[\ddot v_n]_1-3[v_n]_1+4[v_n]_2, \tag{49}$$

$$[v_n]_4 = -h^4[\ddot{v}_n]_2 + 4[v_n]_1 - 7[v_n]_2 + 4[v_n]_3, \quad (50)$$

$$[v_n]_{j+2} = -h^4[\ddot{v}_n]_j - [v_n]_{j-2} + 4[v_n]_{j-1} - 6[v_n]_j + 4[v_n]_{j+1} \qquad \forall\, j \in \{3, 4, \ldots n-2\}, \quad (51)$$

$$h^4[\ddot{v}_n]_{n-1} = -[v_n]_{n-3} + 4[v_n]_{n-2} - 7[v_n]_{n-1} + 4[v_n]_n - \frac{h^3 f_n}{3}, \quad (52)$$

$$h^4[\ddot{v}_n]_n = -[v_n]_{n-2} + 4[v_n]_{n-1} - 3[v_n]_n - \frac{2h^3 f_n}{3}, \quad (53)$$

Here $[v_n]_j$ denotes the j^{th}-component of v_n and we have suppressed the time argument for the sake of brevity. Equations (52) and (53) can be equivalently written as

$$f_n = \frac{1}{h^3}\left([v_n]_n - 3[v_n]_{n-1} + 3[v_n]_{n-2} - [v_n]_{n-3}\right) - h([\ddot{v}_n]_n + [\ddot{v}_n]_{n-1}), \quad (54)$$

$$0 = \frac{1}{6h}\left(11[v_n]_n - 18[v_n]_{n-1} + 9[v_n]_{n-2} - 2[v_n]_{n-3}\right) + \frac{h^3}{6}([\ddot{v}_n]_n - 2[\ddot{v}_n]_{n-1}). \quad (55)$$

Note that (54) is obtained by adding (52) and (53) and (55) is obtained by eliminating f_n from (52) and (53).

Next we derive a parametrization of v_n and f_n in (48) in terms of two functions y_1 and y_2 and their even derivatives. Define

$$y_1 = [v_n]_1, \quad y_2 = 2\,\frac{[v_n]_2 - [v_n]_1}{h^2} \quad (56)$$

so that $[v_n]_1 = y_1$ and $[v_n]_2 = y_1 + h^2 y_2/2$. Using these expressions for $[v_n]_1$ and $[v_n]_2$ and (49) we can express $[v_n]_3$ as a linear combination of y_1, y_2 and their even derivatives as follows:

$$[v_n]_3 = y_1 - h^4 y_1^{(2)} + 2\,h^2 y_2. \quad (57)$$

Similarly using the above expressions for $[v_n]_1$, $[v_n]_2$ and $[v_n]_3$ and (50) we can express $[v_n]_4$ as a linear combination of y_1, y_2 and their even derivatives as follows:

$$[v_n]_4 = y_1 - 5\,h^4 y_1^{(2)} + \frac{9}{2}h^2 y_2 - \frac{1}{2}h^6 y_2^{(2)}. \quad (58)$$

Repeating the above procedure successively for $j = 3, 4, \ldots n-2$, i.e. using the expressions for $[v_n]_{j-2}$, $[v_n]_{j-1}$, $[v_n]_j$ and $[v_n]_{j+1}$ and (51) to express $[v_n]_{j+2}$ as a linear combination of y_1, y_2 and their even derivatives, we obtain the following set of expressions: For $j \in \{1, 2, \ldots n\}$,

$$[v_n]_j = \sum_{k=0}^{\lfloor\frac{j-1}{2}\rfloor} \left[p_{j,k} h^{4k} y_1^{(2k)} + q_{j,k} h^{4k+2} y_2^{(2k)}\right]. \quad (59)$$

Here $\lfloor c \rfloor$ denotes the largest integer less than or equal to $c \in \mathbb{R}$ and $p_{j,k}$ and $q_{j,k}$ are some real coefficients. Finally, substituting the expressions for $[v_n]_n$, $[v_n]_{n-1}$, $[v_n]_{n-2}$ and $[v_n]_{n-3}$ obtained from (59) in (54) and (55) we can rewrite (54) and (55) as

$$f_n = \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \left[a_{n,k} y_1^{(2k)} + b_{n,k} y_2^{(2k)}\right], \quad (60)$$

$$0 = \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \left[c_{n,k} y_1^{(2k)} + d_{n,k} y_2^{(2k)}\right], \quad (61)$$

where $a_{n,k}, b_{n,k}, c_{n,k}$ and $d_{n,k}$ are some real coefficients. Expressions (59)-(61) are the desired parametrization of v_n and f_n in terms of y_1 and y_2.

Choose $y_1, y_2 \in C^\infty[0,T]$ which satisfy (61). Then $[v_n]_j$ for $j \in \{1, 2, \ldots n\}$ and f_n defined using (59) and (60) also belong to $C^\infty[0,T]$ and they satisfy (49)-(55). (This is because the parametrization (59)-(61) has essentially been derived by solving (49)-(55) successively.) Therefore v_n and f_n determined by the chosen y_1 and y_2 satisfy (25) for $t \in [0,T]$. In other words, $\left[\begin{smallmatrix} v_n \\ \dot{v}_n \end{smallmatrix}\right]$ defined using (59) is the solution of (25) on the interval $[0,T]$ corresponding to the input f_n defined using (60) and initial state $\left[\begin{smallmatrix} v_n(0) \\ \dot{v}_n(0) \end{smallmatrix}\right]$ defined using (59). The following remark gives the formula for some of the coefficients appearing in the parametrization.

Remark 2. *For integers $0 \le m \le l$, the binomial coefficient is*

$$\binom{l}{m} = \frac{l!}{m!(l-m)!}. \quad (62)$$

When $m > l$ or $m < 0$, we take

$$\binom{l}{m} = 0. \quad (63)$$

Recall the coefficients $p_{j,k}$ and $q_{j,k}$ from (59). The formulae for these coefficients are

$$p_{j,k} = (-1)^k \binom{j+2k-1}{4k}, \quad (64)$$

$$q_{j,k} = \frac{(-1)^k}{2}\left[\binom{j+2k-1}{4k+2} + \binom{j+2k}{4k+2}\right], \quad (65)$$

for all $j \in \{1, 2, \ldots n\}$ and $k \in \{0, 1, \ldots \lfloor (j-1)/2 \rfloor\}$, see Section B in the Appendix for a proof. Recall $c_{n,0}$ and $d_{n,0}$ from (61). In Section D and Section E in the Appendix we have shown that

$$c_{n,0} = 0, \qquad d_{n,0} = 1. \quad (66)$$

The next theorem is the main result of this section. It addresses the problem of finding a control input f_n which transfers (25) from the zero state to any other steady state over the time interval $[0,T]$. We will use the following function in the theorem: For $t \in [0,T]$ define

$$\psi(t) = \left(\int_0^t \psi_0(\tau)d\tau \Big/ \int_0^T \psi_0(\tau)d\tau\right), \quad (67)$$

where the function $\psi_0 : [0, T] \to \mathbb{R}$ is given as

$$\psi_0(t) = \begin{cases} \exp\left(-\left[\left(1-\frac{t}{T}\right)\frac{t}{T}\right]^{-2}\right), & t \in (0, T) \\ 0, & t=0 \text{ or } t=T \end{cases}.$$

From [23] we get that $\psi \in G_{1.5}[0, T]$ (i.e. ψ is a Gevrey function of order 1.5) and

$$\psi(0) = 0, \quad \psi(T) = 1, \quad \psi^{(k)}(0) = \psi^{(k)}(T) = 0 \tag{68}$$

for all integers $k \geq 1$.

Theorem 3. *Let $T > 0$ and a steady state $\left[\begin{smallmatrix} v_T \\ 0 \end{smallmatrix}\right]$ of the semi-discrete system (25) with $v_T = [\alpha\ \alpha\ \cdots\ \alpha]^\top \in \mathbb{R}^n$ be given. Recall ψ from (67). Define the functions $y, y_1, y_2 \in C^\infty[0, T]$ as follows: For each $t \in [0, T]$, let $y(t) = \alpha\psi(t)$ and*

$$y_1(t) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} d_{n,i} y^{(2i)}(t), \quad y_2(t) = -\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} c_{n,i} y^{(2i)}(t). \tag{69}$$

Then the input $f_n \in C^\infty[0, T]$ defined using (60) transfers the semi-discrete system (25) from the zero state to the steady state $\left[\begin{smallmatrix} v_T \\ 0 \end{smallmatrix}\right]$ over the time interval $[0, T]$, i.e. the solution $\left[\begin{smallmatrix} v_n \\ \dot{v}_n \end{smallmatrix}\right]$ of the semi-discrete system (25) with the initial state $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and input f_n satisfies $\left[\begin{smallmatrix} v_n(T) \\ \dot{v}_n(T) \end{smallmatrix}\right] = \left[\begin{smallmatrix} v_T \\ 0 \end{smallmatrix}\right]$.

Proof: Via a simple substitution it is easy to check that y_1 and y_2 defined in (69) satisfy (61). Hence $\left[\begin{smallmatrix} v_n \\ \dot{v}_n \end{smallmatrix}\right]$ defined using (59) is the solution of (25) on the interval $[0, T]$ corresponding to the input $f_n \in C^\infty[0, T]$ defined using (60) and initial state $\left[\begin{smallmatrix} v_n(0) \\ \dot{v}_n(0) \end{smallmatrix}\right]$ defined using (59) (see the discussion above Remark 2). We will complete the proof of the theorem by verifying that this solution $\left[\begin{smallmatrix} v_n \\ \dot{v}_n \end{smallmatrix}\right]$ has the desired initial and final states, i.e.

$$\begin{bmatrix} v_n(0) \\ \dot{v}_n(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_n(T) \\ \dot{v}_n(T) \end{bmatrix} = \begin{bmatrix} v_T \\ 0 \end{bmatrix}. \tag{70}$$

From the properties of ψ in (68) we get that

$$y(0) = 0, \quad y(T) = \alpha, \quad y^{(k)}(0) = y^{(k)}(T) = 0 \quad \forall\, k \geq 1.$$

Using this in (69) and recalling (66) it follows that the functions y_1 and y_2 satisfy

$$y_1(0) = 0, \quad y_1(T) = \alpha, \quad y_1^{(k)}(0) = y_1^{(k)}(T) = 0 \quad \forall\, k \geq 1,$$
$$y_2(0) = 0, \quad y_2(T) = 0, \quad y_2^{(k)}(0) = y_2^{(k)}(T) = 0 \quad \forall\, k \geq 1.$$

Using the above conditions in (59) to determine the state of the solution $\left[\begin{smallmatrix} v_n \\ \dot{v}_n \end{smallmatrix}\right]$ at times $t = 0$ and $t = T$ we get

$$[v_n]_j(0) = 0, \quad [v_n]_j(T) = \alpha p_{j,0},$$
$$[\dot{v}_n]_j(0) = 0, \quad [\dot{v}_n]_j(T) = 0$$

for all $j \in \{1, 2, \ldots, n\}$. Since $p_{j,0} = 1$, which follows by taking $k = 0$ in (64), we get from the above conditions that the solution $\left[\begin{smallmatrix} v_n \\ \dot{v}_n \end{smallmatrix}\right]$ satisfy (70). This completes the proof of the theorem. $\square$

A. NUMERICAL SIMULATIONS

In this section we numerically illustrate our solution to the motion planning problem described in Theorem 3. In our simulations we take $n = 8$ and the goal is to transfer (25) from the zero state to the steady state $\left[\begin{smallmatrix} v_T \\ 0 \end{smallmatrix}\right]$ with $v_T = [1\ 1\ \cdots\ 1]^\top \in \mathbb{R}^8$ over the time interval $[0, 5]$. We have computed the control input f_n required for this transfer by following the procedure in the statement of the theorem. Figures 2 and 3 show the solution $\left[\begin{smallmatrix} v_n \\ \dot{v}_n \end{smallmatrix}\right]$ of (25) for this input and zero initial state. As seen from the plots, the solution starts at the zero state at 0 seconds and reaches the desired steady state at 5 seconds.

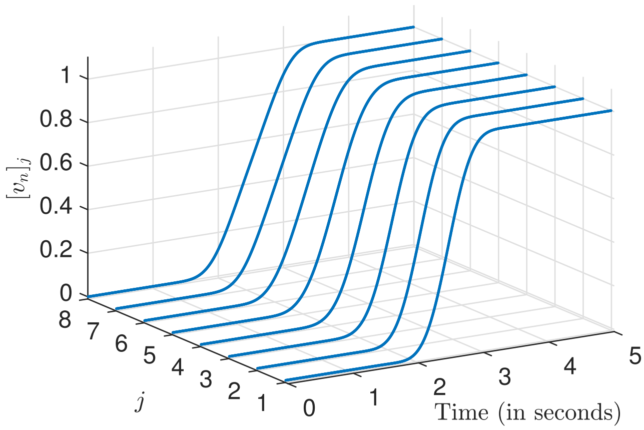

FIGURE 2. The first component v_n of the solution of (25) starts from $[0\ 0\ \cdots\ 0]^\top \in \mathbb{R}^8$ at 0 seconds and reaches $[1\ 1\ \cdots\ 1]^\top \in \mathbb{R}^8$ at 5 seconds as desired.

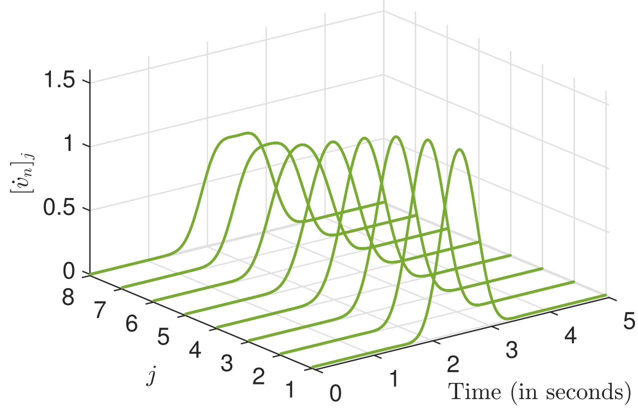

FIGURE 3. The second component $\dot{v}_n$ of the solution of (25) starts from $[0\ 0\ \cdots\ 0]^\top \in \mathbb{R}^8$ at 0 seconds and goes back to $[0\ 0\ \cdots\ 0]^\top \in \mathbb{R}^8$ at 5 seconds as desired.

V. MOTION PLANNING FOR THE BEAM PDE WITH CONSTANT *EI*

In this section we take $EI = 1$ (like in Section IV) and present our solution to the motion planning problem, Problem 1, for the PDE (1)-(3) by building on our results in Sections III and IV. More specifically, fix $T > 0$ and suppose that $[w_T\ \ 0]^\top$ is a steady state of (1)-(3) with $w_T(x) = \alpha$ for some $\alpha \in \mathbb{R}$ and all $x \in [0, 1]$. Then $\left[\begin{smallmatrix} v_T \\ 0 \end{smallmatrix}\right]$, where $v_T = [\alpha\ \alpha\ \cdots\ \alpha]^\top \in \mathbb{R}^n$, is a steady state for the n^{th}-order semi-discrete system (25). Let f_n be the control input in Theorem 3 which transfers (25) from the zero initial state to the steady state $\left[\begin{smallmatrix} v_T \\ 0 \end{smallmatrix}\right]$ over the time interval $[0, T]$. Observe that by substituting the expressions for y_1 and y_2 from (69) into (60) we can write f_n as

$$f_n = \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor} a_{n,k} d_{n,i} y^{(2k+2i)} - \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor} b_{n,k} c_{n,i} y^{(2k+2i)}, \tag{71}$$

where y is as in Theorem 3. Define

$$a_{n,k} = b_{n,k} = c_{n,k} = d_{n,k} = 0 \quad \forall\, k > \lfloor (n+1)/2 \rfloor. \tag{72}$$

Using this and changing the order of the summations in (71) we can rewrite f_n as

$$f_n = \sum_{k=0}^{\infty} r_{n,k} y^{(2k)}, \tag{73}$$

where

$$r_{n,k} = \sum_{\substack{i+l=k \\ i,l \geq 0}} (a_{n,i} d_{n,l} - b_{n,i} c_{n,l}). \tag{74}$$

In Proposition 4 we show that $\lim_{n\to\infty} r_{n,k} = r_k$ for some $r_k \in \mathbb{R}$ and all $k \geq 0$ and in Proposition 5 we derive some bounds for $r_{n,k}$ and r_k. Using these results we prove in Proposition 6 that the sequence of control inputs $\{f_n\}_{n=5}^{\infty}$ converges to a limit f. In Theorem 7 we show that this f transfers the PDE (1)-(3) from the zero initial state to the steady state $[w_T \;\; 0]^\top$ over the time interval $[0, T]$, i.e. f solves Problem 1.

Proposition 4. *For each $k \geq 0$,*

$$\lim_{n\to\infty} r_{n,k} = r_k \tag{75}$$

for some $r_k \in \mathbb{R}$.

Proof: Observe that $r_{n,k}$ in (74) is given by a finite sum. We claim that

$$\lim_{n\to\infty} a_{n,0} = \lim_{n\to\infty} b_{n,0} = \lim_{n\to\infty} c_{n,0} = 0, \quad \lim_{n\to\infty} d_{n,0} = 1, \tag{76}$$

and for each $k \geq 1$

$$\lim_{n\to\infty} a_{n,k} = \frac{(-1)^k}{(4k-3)!}, \quad \lim_{n\to\infty} b_{n,k} = \frac{(-1)^k}{(4k-1)!}, \tag{77}$$

$$\lim_{n\to\infty} c_{n,k} = \frac{(-1)^k}{(4k-1)!}, \quad \lim_{n\to\infty} d_{n,k} = \frac{(-1)^k}{(4k+1)!}. \tag{78}$$

From the limits in (76), (77), (78) and the definition of $r_{n,k}$ in (74), the limit in (75) follows immediately for each $k \geq 0$. We will complete the proof of this proposition by establishing the limits in (76)-(78).

For $n \geq 5$ we have $a_{n,0} = b_{n,0} = c_{n,0} = 0$ and $d_{n,0} = 1$, see Sections D and E in the Appendix for a proof, and so the limits in (76) hold trivially. We will now establish the first limit in (77) corresponding to $a_{n,k}$. From the procedure used to obtain (60), i.e. substituting the expressions for $[v_n]_n$, $[v_n]_{n-1}$, $[v_n]_{n-2}$ and $[v_n]_{n-3}$ from (59) into (54), it is evident that

$$a_{n,k} = h^{4k-3} \left(p_{n,k} - 3p_{n-1,k} + 3p_{n-2,k} - p_{n-3,k}\right) - h^{4k-3} \left(p_{n,k-1} + p_{n-1,k-1}\right) \tag{79}$$

for all $k \in \{0, 1, \ldots \lfloor (n+1)/2 \rfloor\}$. Here the coefficients $p_{n,k}$, $p_{n-1,k}$, $p_{n-2,k}$, $p_{n-3,k}$, $p_{n,k-1}$ and $p_{n-1,k-1}$ are given by the formula in (64). Using the identity (A.3) with $l = n + 2k - 4$ and $m = 4k$ one can verify using the formula in (64) that

$$p_{n,k} - 3p_{n-1,k} + 3p_{n-2,k} - p_{n-3,k} = (-1)^k \binom{n+2k-4}{4k-3}.$$

Replacing the terms on the right side of (79) using the above formula and the formulae for $p_{n,k-1}$ and $p_{n-1,k-1}$ obtained from (64) we get

$$a_{n,k} = (-1)^k h^{4k-3} \left[\binom{n+2k-4}{4k-3} + \binom{n+2k-3}{4k-4} + \binom{n+2k-4}{4k-4}\right]. \tag{80}$$

Writing the binomial coefficients in the above expression in terms of factorials and recalling that $h = 1/(n-1)$ we can rewrite (80) as follows:

$$a_{n,k} = (-1)^k \left[\frac{1}{(4k-3)!} \prod_{i=0}^{4k-4} \frac{n+2k-4-i}{n-1}, + \frac{1}{(n-1)(4k-4)!} \prod_{i=0}^{4k-5} \frac{n+2k-3-i}{n-1} + \frac{1}{(n-1)(4k-4)!} \prod_{i=0}^{4k-5} \frac{n+2k-4-i}{n-1}\right]. \tag{81}$$

The three products denoted by the $\prod$ sign in the above equation converge to 1 as $n \to \infty$ and the terms multiplying the second and third among them converge to zero as $n \to \infty$. Taking the limit as $n \to \infty$ on both sides of the above equation the first limit in (77) corresponding to $a_{n,k}$ follows. The limits corresponding to $b_{n,k}$, $c_{n,k}$ and $d_{n,k}$ in (77) and (78) can be established similarly. ☐

In the last proposition we showed that $r_{n,k}$ converges to r_k as $n \to \infty$. We derive certain bounds for $r_{n,k}$ and r_k in the following proposition. In the proof below, the factorial of negative integers must be understood to be 1.

Proposition 5. *There exists a constant $M > 0$ such that*

$$|r_{n,k}| \leq \frac{M^k}{(4k-8)!} \qquad \forall\, k \geq 2, \quad \forall\, n \geq 5, \tag{82}$$

$$|r_k| \leq \frac{M^k}{(4k-8)!} \qquad \forall\, k \geq 2. \tag{83}$$

Proof: We claim that there exists a constant $M_1 > 1$ such that for all $n \geq 5$ and $k \geq 1$ we have

$$|a_{n,k}| \leq \frac{M_1^k}{(4k-4)!}, \quad |b_{n,k}| \leq \frac{M_1^k}{(4k-4)!}, \tag{84}$$

$$|c_{n,k}| \leq \frac{M_1^k}{(4k-4)!}, \quad |d_{n,k}| \leq \frac{M_1^k}{(4k-4)!}. \tag{85}$$

Indeed, the above estimates hold trivially for any $n \geq 5$ and $k > \lfloor (n+1)/2 \rfloor$, see (72). Suppose that $n \geq 5$ and $k \in \{1, 2, \ldots \lfloor (n+1)/2 \rfloor\}$. Then $2k \leq n+1$ and so each term (ratio) in the three products denoted by the $\prod$ sign in (81) is less than 4. Since each of these three products contains less than $4k$ terms and the term in front of these products is less than $1/(4k-4)!$, it follows that $|a_{n,k}| \leq 3(4^{4k})/(4k-4)!$. So the estimate for $a_{n,k}$ in (84) holds. The estimates for $b_{n,k}$, $c_{n,k}$ and $d_{n,k}$ in (84)-(85) can be established similarly. This completes the proof of the claim.

For $n \geq 5$ we have $a_{n,0} = b_{n,0} = c_{n,0} = 0$ and $d_{n,0} = 1$, see Sections D and E in the Appendix. Using this and the estimates in (84)-(85) to bound $r_{n,k}$ in (74) we get

$$|r_{n,k}| \leq \sum_{\substack{i+j=k \\ i,j \geq 0}} \frac{2M_1^k}{(4i-4)!(4j-4)!} \qquad \forall\, k \geq 1 \quad \forall\, n \geq 5.$$

Note that the $(\alpha+1)^{\text{th}}$-term in the binomial expansion of $(1+1)^{\alpha+\beta}$ is less than $2^{\alpha+\beta}$ and so we have $(\alpha+\beta)! \leq 2^{\alpha+\beta}\alpha!\beta!$ for all integers $\alpha, \beta \geq 0$. Using this to bound the terms in the above summation we get that

$$|r_{n,k}| \leq (k+1)\frac{2^{4k+1}M_1^k}{(4k-8)!} \qquad \forall\, k \geq 2 \quad \forall\, n \geq 5,$$

which implies the estimate in (82). The estimate in (83) follows immediately from (75) and (82). □

Denote the limits in (76)-(78) to which $a_{n,k}$, $b_{n,k}$, $c_{n,k}$ and $d_{n,k}$ converge as n tends to infinity by a_k, b_k, c_k and d_k, respectively. Taking the limit as n tends to infinity on both sides of (74) we get

$$r_k = \sum_{\substack{i+l=k \\ i,l \geq 0}} (a_i d_l - b_i c_l) \qquad \forall\, k \geq 0. \tag{86}$$

In the next proposition we will show that the functions f_n defined in (73) converge to the function

$$f = \sum_{k=0}^{\infty} r_k y^{(2k)} \tag{87}$$

when $y \in G_{1.5}[0, T]$.

Proposition 6. *Fix $T > 0$ and $y \in G_{1.5}[0, T]$. Let f and f_n be the functions defined in (87) and (73), respectively. Then $f \in C^\infty[0, T]$ and*

$$\lim_{n\to\infty} \|f_n - f\|_{C^4[0,T]} = 0. \tag{88}$$

Proof: We will first show that $f \in C^\infty[0, T]$ and its i^{th} derivative is

$$f^{(i)}(t) = \sum_{k=0}^{\infty} r_k y^{(2k+i)}(t) \qquad \forall\, t \in [0, T] \tag{89}$$

and for each integer $i \geq 0$. Note that the above series is obtained by termwise differentiation of the series in (87). Since $y \in G_{1.5}[0, T]$, there exists a constant $D > 0$ such that

$$\sup_{t\in[0,T]} |y^{(2k)}(t)| \leq D^{2k+1}((2k)!)^{1.5} \qquad \forall\, k \geq 0. \tag{90}$$

Using the estimates in (83) and (90) we get

$$\sup_{t\in[0,T]} \left|r_k y^{(2k+i)}(t)\right| \leq C_k^i, \tag{91}$$

where

$$C_k^i = \frac{M^k D^{2k+1+i}((2k+i)!)^{1.5}}{(4k-8)!}.$$

It is easy to verify that

$$\lim_{k\to\infty} \frac{C_{k+1}^i}{C_k^i} = 0$$

and therefore it follows via the ratio test that the series $\sum_{k\geq 0} C_k^i$ converges for each $i \geq 0$. Since the terms in the series in (89) are bounded by C_k^i (see (91)), we can conclude via the Weierstrass M-test that this series converges uniformly on $[0, T]$ for each $i \geq 0$. Moreover, the limit of this series belongs to $C[0, T]$ since each term in it belongs to $C[0, T]$. In summary, the series in (89) obtained by termwise differentiation of (87) converges uniformly to a continuous function for each $i \geq 0$. This implies that $f \in C^\infty[0, T]$ and its i^{th}-derivative is indeed given by the series in (89).

We will now complete the proof of this proposition by showing that (88) holds. From (73) and (87) we get

$$f_n^{(i)} - f^{(i)} = \sum_{k=0}^{\infty} (r_{n,k} - r_k) y^{(2k+i)}$$

for every $i \geq 0$. Using (82), (83) and (90) it is easy to see that $\|(r_{n,k} - r_k)y^{(2k+i)}\|_{C[0,T]} \leq 2C_k^i$. Since $\sum_{k\geq 0} C_k^i < \infty$ (see the discussion in the paragraph above), it immediately follows that for each $\epsilon > 0$ there exists a $k(\epsilon) > 0$ independent of n such that

$$\left\| \sum_{k=k(\epsilon)}^{\infty} (r_{n,k} - r_k) y^{(2k+i)} \right\|_{C[0,T]} < \epsilon \qquad \forall\, n \geq 5.$$

Furthermore, from (75) and (90) it follows that there exists an $n(\epsilon) > 0$ such that

$$\left\| \sum_{k=0}^{k(\epsilon)-1} (r_{n,k} - r_k) y^{(2k+i)} \right\|_{C[0,T]} < \epsilon \qquad \forall\, n > n(\epsilon).$$

Since ϵ in the above two estimates are arbitrary, we can conclude that $\lim_{n\to\infty} \|f_n^{(i)} - f^{(i)}\|_{C[0,T]} = 0$ for each $i \geq 0$ and so (88) holds. □

Recall the definition and properties of the steady states of the PDE (1)-(3) from Section II. The following theorem presents our solution to the PDE motion planning problem stated in Problem 1.

Theorem 7. *Let $T > 0$ and a steady state $[w_T \;\; 0]^\top$ of the PDE (1)-(3), with $w_T(x) = \alpha$ for some constant $\alpha \in \mathbb{R}$ and* *all $x \in [0, 1]$, be given. Fix $y = \alpha\psi$, where $\psi \in G_{1.5}[0, T]$ is the function in (67). Define the function f via (87), i.e.*

$$f = \sum_{k=0}^{\infty} r_k y^{(2k)},$$

so that $f \in C^\infty[0, T]$ and (16) holds. Then the solution w of the PDE (1)-(3) for the zero initial state and control input f satisfies

$$w(x, T) = w_T \quad \text{and} \quad w_t(x, T) = 0 \quad \forall\, x \in [0, 1], \qquad (92)$$

i.e. the control input f transfers the PDE (1)-(3) from the zero state to the steady state $[w_T \;\; 0]^\top$ over the time interval $[0, T]$. In other words, f solves Problem 1 for the given T and steady state $[w_T \;\; 0]^\top$.

Proof: Recall the discretization operator R_n from the notations in Section I and observe that $\begin{bmatrix} R_n w_T \\ 0 \end{bmatrix}$ is a steady state of the n^{th}-order semi-discrete system (25). Let the function y be as in the theorem statement and define f_n using (73). Let v_n be the solution of the n^{th}-order semi-discrete system (25) for the zero initial state and input f_n. It then follows from Theorem 3 that

$$v_n(T) = R_n w_T, \quad \dot{v}_n(T) = 0, \qquad (93)$$

i.e. f_n transfers (25) from the zero state to the steady state $\begin{bmatrix} R_n w_T \\ 0 \end{bmatrix}$ over the time interval $[0, T]$.

From the properties of ψ in (68) and since $y = \alpha\psi$, we get that $y \in C^\infty[0, T]$ and

$$y^{(i)}(0) = 0 \qquad \forall\, i \geq 0. \qquad (94)$$

Recall that f_n is also given by the finite sum in (71) and so clearly $f_n \in C^\infty[0, T]$ and $f_n^{(i)}(0) = 0$ for all $i \geq 0$. From Proposition 6 we get that f defined via (87) belongs to $C^\infty[0, T]$ and $\lim_{n\to\infty} \|f_n - f\|_{C^4[0,T]} = 0$. Furthermore, using (89) and (94) we get that $f^{(i)}(0) = 0$ for all $i \geq 0$. In summary, the inputs f_n and f satisfy all the hypothesis in Proposition 1 and Theorem 2.

From Proposition 1 we get that the solution w of the PDE (1)-(3) corresponding to zero initial state and input f belongs to $C^\infty([0, T]; C^5[0, 1])$. From Theorem 2 it follows that this w and v_n defined above (93) satisfy

$$\lim_{n\to\infty} \|R_n w(\cdot, T) - v_n(T)\|_\infty = 0,$$
$$\lim_{n\to\infty} \|R_n w_t(\cdot, T) - \dot{v}_n(T)\|_\infty = 0.$$

This, together with (93), gives

$$\lim_{n\to\infty} \left(\|R_n w(\cdot, T) - R_n w_T\|_\infty + \|R_n w_t(\cdot, T)\|_\infty\right) = 0.$$

Now since $w(\cdot, T), w_t(\cdot, T), w_T \in C[0, 1]$, we can conclude from the above expression that (92) holds. □

Remark 3. *Consider an Euler-Bernoulli beam PDE with boundary input. The n^{th}-order semi-discrete systems, derived using the standard finite-difference scheme, for approximating the adjoint system of this beam PDE are often not uniformly (in n) observable. This anomaly is caused by the behaviour of the high frequency eigenvalues of the semi-discrete systems. It can however be rectified by modifying the semi-discrete systems using filtering [8], [9], [11], and the averaging operator [10], [12], so that the modified semi-discrete systems are uniformly (in n) observable. Hence suitably modified n^{th}-order semi-discrete approximations of the beam PDE can be used to construct control inputs for transferring the beam PDE between any two given states, see [8], [9], and [10]. The current paper is significantly different from these works for two reasons: (i) In this paper we restrict our attention to the practically relevant motion planning problem of transferring the beam PDE between steady states, whereas [8], [9], and [10] consider the more general problem of transferring the beam PDE between arbitrary states in the state space, and (ii) we use the flatness technique to construct our control inputs which is different from the techniques used in [8], [9], and [10]. For these reasons, unlike [8], [9], and [10], we can use the n^{th}-order semi-discrete systems obtained using the standard finite-difference scheme without modifications to construct the desired control signals, and we have established this rigorously in this section. Finally, we remark that unlike [8] which considers clamped boundary conditions, [9] which considers hinged boundary conditions and [10] which considers cantilever boundary conditions, we consider sliding cantilever boundary conditions in this paper.*

A. NUMERICAL SIMULATIONS

In this section we numerically illustrate our solution to the motion planning problem, Problem 1, for the beam PDE (1)-(3) presented in Theorem 7.

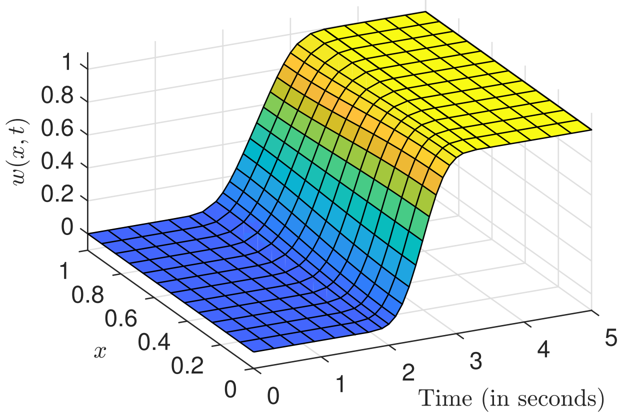

FIGURE 4. The trajectory of the beam displacement w starts from $w(\cdot, 0) = 0$ at 0 seconds and reaches $w(\cdot, 5) = 1$ at 5 seconds as desired.

In our simulations, the goal is to transfer (1)-(3) from the zero state to the steady state $[w_T \;\; 0]^\top$, with $w_T(x) = 1$ for all $x \in [0, 1]$, over the time interval $[0, 5]$. We have computed the control input f required for this transfer by following the procedure in the statement of the theorem. Observe that the expression for f is an infinite series, see (87). We truncate this series after 21 terms to compute a sufficiently accurate approximation of f. Figure 4 shows the solution w of (1)-(3) for this input and zero initial state and Figure 5 shows the time derivative w_t. As seen from the plots, the solution starts at the zero state at 0 seconds and reaches the desired steady state at 5 seconds.

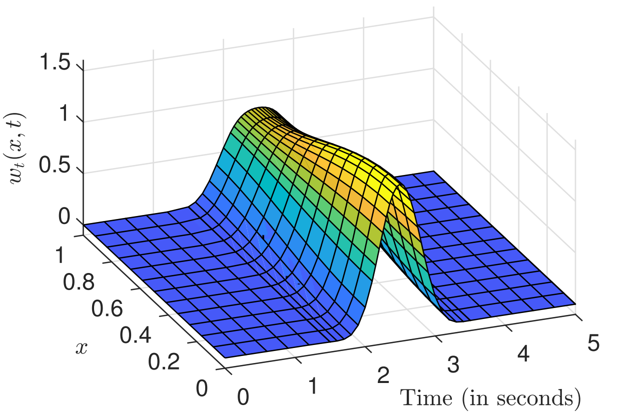

FIGURE 5. The trajectory of the beam velocity w_t starts from $w_t(\cdot, 0) = 0$ at 0 seconds and goes back to $w_t(\cdot, 5) = 0$ at 5 seconds as desired.

VI. MOTION PLANNING FOR THE NONUNIFORM BEAM PDE

In this section we describe a procedure for constructing control inputs which can solve the motion planning problem, Problem 1, even when the flexural rigidity of the beam PDE is spatially varying. This procedure is a generalization of the approach that we have presented in Sections IV and V for the uniform beam. However, unlike in Sections IV and V we do not provide any proofs in this section. Instead we present a step-by-step guideline for constructing the control input and illustrate the efficacy of our procedure using an example.

We will first derive parameterizations for the state and input of the n^{th}-order semi-discrete approximation (25). These parameterizations, which are analogous to those developed in Section IV for the uniform beam, will be required for constructing the desired control inputs using our guideline.

Note that the n^{th}-order semi-discrete system for the non-uniform beam (25) can equivalently be written as

$$\ddot{v}_n(t) = -P_n v_n(t) + B_n f_n(t) \qquad \forall\, t \geq 0, \tag{95}$$

which is a collection of n scalar differential equations. Here $P_n = L_n^\top E_n L_n$. By rearranging the terms in each of these equations we obtain the following set of n equations which are equivalent to (25):

$$[v_n]_3 = h^4\sigma_{1,0}[\ddot{v}_n]_1 + \sigma_{1,1}[v_n]_1 + \sigma_{1,2}[v_n]_2, \tag{96}$$

$$[v_n]_4 = h^4\sigma_{2,0}[\ddot{v}_n]_2 + \sigma_{2,1}[v_n]_1 + \sigma_{2,2}[v_n]_2 + \sigma_{2,3}[v_n]_3, \tag{97}$$

$$[v_n]_{j+2} = h^4\sigma_{j,0}[\ddot{v}_n]_j + \sigma_{j,j-2}[v_n]_{j-2} + \sigma_{j,j-1}[v_n]_{j-1} + \sigma_{j,j}[v_n]_j + \sigma_{j,j+1}[v_n]_{j+1} \quad \forall\, j \in \{3, 4, \dots n-2\}, \tag{98}$$

$$h^4[\ddot{v}_n]_{n-1} = \sigma_{n-1,n-3}[v_n]_{n-3} + \sigma_{n-1,n-2}[v_n]_{n-2} + \sigma_{n-1,n-1}[v_n]_{n-1} + \sigma_{n-1,n}[v_n]_n - \frac{h^3 f_n}{3}, \tag{99}$$

$$h^4[\ddot{v}_n]_n = \sigma_{n,n-2}[v_n]_{n-2} + \sigma_{n,n-1}[v_n]_{n-1} + \sigma_{n,n}[v_n]_n - \frac{2h^3 f_n}{3}. \tag{100}$$

The above equations are analogous to (49)-(53) derived in Section IV for the uniform beam. The constants $\sigma_{i,j}$ in these equations are determined by the flexural rigidity EI, see Appendix C for the definition of these constants. Equations (99) and (100) can be equivalently written as:

$$f_n = \frac{1}{h^3}\Big(\sigma_{n-1,n-3}[v_n]_{n-3} + (\sigma_{n-1,n-2} + \sigma_{n,n-2})[v_n]_{n-2} + (\sigma_{n-1,n-1} + \sigma_{n,n-1})[v_n]_{n-1} + (\sigma_{n-1,n} + \sigma_{n,n})[v_n]_n\Big) - h([\ddot{v}_n]_{n-1} + [\ddot{v}_n]_n) \tag{101}$$

$$0 = \frac{1}{6h}\Big(2\sigma_{n-1,n-3}[v_n]_{n-3} + (2\sigma_{n-1,n-2} - \sigma_{n,n-2})[v_n]_{n-2} + (2\sigma_{n-1,n-1} - \sigma_{n,n-1})[v_n]_{n-1} + (2\sigma_{n-1,n} - \sigma_{n,n})[v_n]_n\Big) - \frac{h^3}{6}(2[\ddot{v}_n]_{n-1} - [\ddot{v}_n]_n). \tag{102}$$

Note that (101) is obtained by adding (99) and (100) and (102) is obtained by eliminating f_n from (99) and (100). Define

$$y_1 = [v_n]_1, \quad y_2 = 2\,\frac{[v_n]_2 - [v_n]_1}{h^2}. \tag{103}$$

Following the procedure used to obtain (59), (60) and (61) from (49)-(55) by substituting for $[v_n]_1$ and $[v_n]_2$ from (56), we obtain (104), (105) and (106) from (96)-(102) by substituting for $[v_n]_1$ and $[v_n]_2$ from (103):

$$[v_n]_j = \sum_{k=0}^{\lfloor \frac{j-1}{2} \rfloor}\Big[p_{j,k}h^{4k}y_1^{(2k)} + q_{j,k}h^{4k+2}y_2^{(2k)}\Big]. \tag{104}$$

$$f_n = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor}\Big[a_{n,k}y_1^{(2k)} + b_{n,k}y_2^{(2k)}\Big], \tag{105}$$

$$0 = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor}\Big[c_{n,k}y_1^{(2k)} + d_{n,k}y_2^{(2k)}\Big], \tag{106}$$

Once n is fixed, the procedure used to obtain (104)-(106) naturally yields the values of the coefficients $p_{j,k}$, $q_{j,k}$, $a_{n,k}$, $b_{n,k}$, $c_{n,k}$ and $d_{n,k}$.

Suppose that $y_1, y_2 \in C^\infty[0,T]$ are chosen such that (106) holds. Then v_n and f_n determined via (104) and (105) solve (95), see the discussion below (61) for the reasoning. So (104)-(106) is a parameterization for the n^{th}-order semi-discrete approximation (25) of the nonuniform beam. We now present our guideline for constructing control inputs which can solve the motion planning problem, Problem 1.

Guideline for solving Problem 1

Given. A steady state $[w_T \quad 0]^\top$ of the PDE (1)-(3) with $w_T(x) = \alpha$ for all $x \in [0, 1]$ and some $\alpha \in \mathbb{R}$ and a time of transfer $T > 0$.

Step 1. Fix $n \in \mathbb{N}$ large. Find the values of the coefficients $a_{n,k}$, $b_{n,k}$, $c_{n,k}$ and $d_{n,k}$ appearing in (105)-(106).

Step 2. Let $y = \frac{\alpha\psi}{d_{n,0}}$, where ψ is the function defined in (67). Compute the functions y_1 and y_2 as follows:

$$y_1(t) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} d_{n,k} y^{(2k)}(t), \quad y_2(t) = -\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} c_{n,k} y^{(2k)}(t)$$

for each $t \in [0, T]$.

Step 3. Compute the control input f_n using (105) and the functions y_1 and y_2 obtained from Step 2.

Step 4. Apply f_n to the PDE (1)-(3) and verify if $\|w(\cdot, T) - w_T\|_{C^1[0,1]}$ is sufficiently small. If not, increase n and repeat Steps 1-4.

The control input f_n obtained using the above guideline transfers the PDE satisfactorily from the zero initial state to the desired final steady provided n is sufficiently large. Moreover, f_n converges to a limiting function f as $n \to \infty$ which solves Problem 1. We have verified this extensively in simulations (one such simulation is presented in the next section) and hope to establish it theoretically in a future work. We remark that the above guideline for constructing the control inputs is consistent with the approach presented in Sections IV and V for constructing control inputs for the uniform beam. In particular, when $EI = 1$, the input f_n computed using the guideline converges to the control input f presented in Section V as $n \to \infty$.

Remark 4. *In this paper we have presented an approach for constructing control signals which can transfer Euler-Bernoulli beams with sliding cantilever boundary conditions between steady states. We have established the efficacy of this approach theoretically for uniform beams in Sections IV and V and via a numerical example for nonuniform beams in this section. Our approach can in principle be used for transferring Euler-Bernoulli beams with other boundary conditions between steady states. Indeed, in our conference paper [21] we have demonstrated this numerically for Euler-Bernoulli beams with hinged boundary conditions.*

A. NUMERICAL SIMULATIONS

In this section we consider a motion planning problem, Problem 1, for a non-uniform beam PDE. We demonstrate numerically that the control inputs f_n constructed for this problem using the guideline given above converge to a limiting function f as $n \to \infty$ and that f_n solves the problem satisfactorily provided n is sufficiently large.

Consider the beam PDE (1)-(3) with the following flexural rigidity: $EI(x) = 1 + 3x$ for all $x \in [0, 1]$. Our objective is to solve the motion planning problem of transferring this beam PDE from the zero initial state to the final steady state $[w_T \quad 0]^\top$, where $w_T(x) = 1$ for all $x \in [0, 1]$, over a 5 second time interval. Accordingly, for $n \in \{10, 100, 200, \ldots 1000\}$, we compute the control input f_n required for this transfer by following the steps in our guideline and verify numerically that the control input f_{1000} solves the motion planning problem satisfactorily. Figure 6 shows that the difference between f_n and f_m is small when n and m are large, which indicates that f_n converges to a limiting function f as $n \to \infty$. This is also evident from the plots of f_n for $n \in \{10, 30, 50, 70, 100, 1000\}$ shown in Figure 7. Figures 8

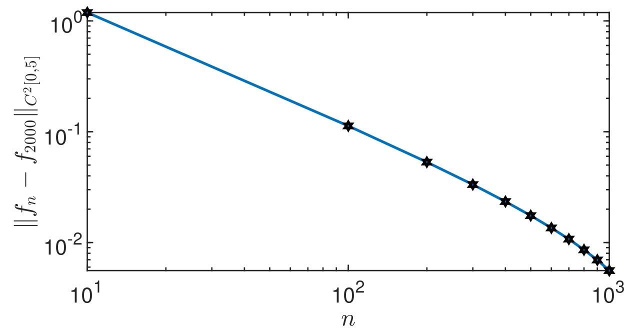

FIGURE 6. The difference $\|f_n - f_{2000}\|_{C^2[0,5]}$ decays as n is increased which indicates that f_n converges to a limiting function f as $n \to \infty$.

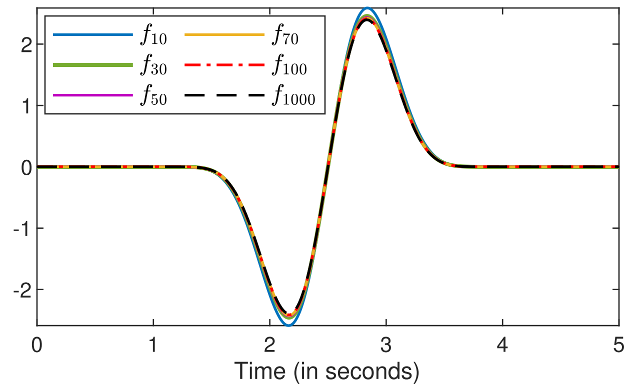

FIGURE 7. The plots of f_n for $n \in \{10, 100, 200, \ldots 1000\}$. It is evident from these plots that f_n converges to a limiting function f as $n \to \infty$.

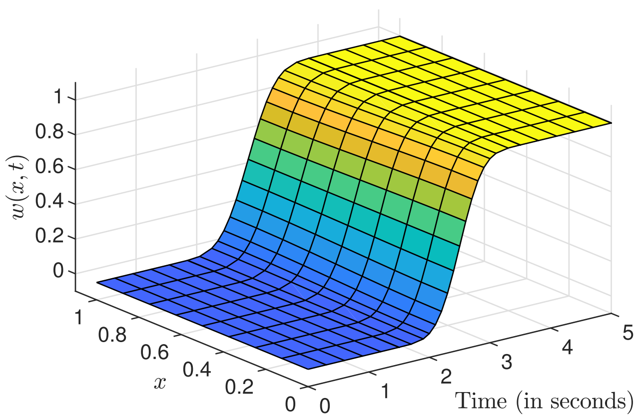

FIGURE 8. The trajectory of the beam displacement w starts from zero at 0 seconds and reaches close to one at 5 seconds when the control input is f_{1000}.

and 9 show that f_{1000} transfers the PDE (1)-(3) close to the final steady state $[w_T \;\; 0]^\top$ in 5 seconds.

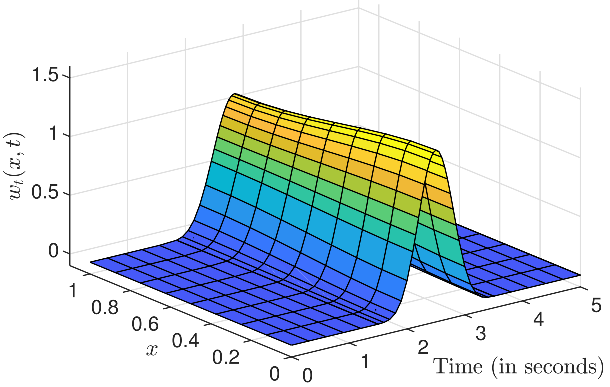

FIGURE 9. **The trajectory of the beam velocity w_t starts from zero at 0 seconds and comes back close to zero at 5 seconds when the control input is f_{1000}.**

VII. CONCLUSION

In this paper we have solved a motion planning problem for a Euler-Bernoulli beam with sliding cantilever boundary conditions using the early lumping approach. We have derived a finite-difference based semi-discrete approximation of the beam PDE and showed that the solution to the semi-discrete system converges to the solution of the PDE as the discretization step size tends to zero. Given a steady state of the beam PDE and a time interval, in the case of a uniform beam we have shown that control signals derived via the flatness technique for transferring the semi-discrete system from the zero state to the steady-state obtained by discretizing the given beam steady-state converge to a limiting control signal as the discretization step size tends to zero, and that this limiting control signal transfers the beam PDE from the zero state to the desired steady state over the given time interval. The same is also true in the case of a nonuniform beam and we have demonstrated this using a numerical example. Future work will focus on extending the early lumping approach to solve motion planning problems for more complex (nonlinear, higher-dimensional) PDEs.

APPENDIX

A. PROPERTIES OF THE BINOMIAL COEFFICIENTS

Recall the binomial coefficients from Remark 2, see (62) and (63). We present four identities below which we will use in this Appendix. Let l and m be two integers. The first identity is

$$\binom{l+1}{m} - \binom{l}{m} = \binom{l}{m-1} \qquad \forall\, l \geq 0, \qquad \text{(A.1)}$$

see [24, Chapter 5,Section 1] for a proof. Substituting $l+1$ in place of l in the above identity we get

$$\binom{l+2}{m} - \binom{l+1}{m} = \binom{l+1}{m-1}.$$

Subtracting (A.1) from the above equation and simplifying the terms on the right side of the resulting equation, by using (A.1) with $m-1$ in place of m, we get our second identity:

$$\binom{l+2}{m} - 2\binom{l+1}{m} + \binom{l}{m} = \binom{l+1}{m-1} - \binom{l}{m-1} = \binom{l}{m-2} \quad \forall\, l \geq 0. \quad \text{(A.2)}$$

Using $l+1$ in place of l in the above equation we get

$$\binom{l+3}{m} - 2\binom{l+2}{m} + \binom{l+1}{m} = \binom{l+1}{m-2}.$$

Subtracting (A.2) from the above equation and simplifying the terms on the right side of the resulting equation, by using (A.1) with $m-2$ in place of m, we get our third identity: For any $l \geq 0$,

$$\binom{l+3}{m} - 3\binom{l+2}{m} + 3\binom{l+1}{m} - \binom{l}{m} = \binom{l}{m-3}. \qquad \text{(A.3)}$$

Proceeding this way, i.e. subtracting (A.3) from the expression obtained by replacing l with $l+1$ in (A.3) and then using (A.1) to simplify the terms on the right side we get our fourth and final identity:

$$\binom{l+4}{m} - 4\binom{l+3}{m} + 6\binom{l+2}{m} - 4\binom{l+1}{m} + \binom{l}{m} = \binom{l}{m-4} \quad \forall\, l \geq 0. \quad \text{(A.4)}$$

B. VERIFICATION OF THE FORMULAE FOR $p_{j,k}$ AND $q_{j,k}$

The parametrization for v_n in (59) can equivalently be written as

$$[v_n]_j = \sum_{k=0}^{n} \left[p_{j,k} h^{4k} y_1^{(2k)} + q_{j,k} h^{4k+2} y_2^{(2k)} \right] \qquad \text{(A.5)}$$

if we define $p_{j,k} = q_{j,k} = 0$ for $j \in \{1, 2, \ldots n\}$ and $k \in \{\lfloor (j-1)/2 \rfloor + 1, \lfloor (j-1)/2 \rfloor + 2, \ldots n\}$. Using the convention adopted in (63) it is easy to see that the $p_{j,k}$ and $q_{j,k}$ for these values of j and k are also given by the formulae (64) and (65). We verify below that (A.5) holds with $p_{j,k}$ and $q_{j,k}$ given by the formulae in (64) and (65) for $j \in \{1, 2, \ldots n\}$ and $k \in \{0, 1, \ldots n\}$.

Comparing the parameterizations for $[v_n]_1$, $[v_n]_2$, $[v_n]_3$ and $[v_n]_4$ obtained from (A.5) with (56), (57) and (58) it is easy to see that

$$p_{1,0} = 1, \quad p_{1,k} = 0 \quad \forall\, k \in \{1, 2, \ldots n\},$$
$$q_{1,k} = 0 \quad \forall\, k \in \{0, 1, \ldots n\},$$
$$p_{2,0} = 1, \quad p_{2,k} = 0 \quad \forall\, k \in \{1, 2, \ldots n\},$$
$$q_{2,0} = \frac{1}{2}, \quad q_{2,k} = 0 \quad \forall\, k \in \{1, 2, \ldots n\},$$
$$p_{3,0} = 1, \quad p_{3,1} = -1, \quad p_{3,k} = 0 \quad \forall\, k \in \{2, 3, \ldots n\},$$
$$q_{3,0} = 2, \quad q_{3,k} = 0 \quad \forall\, k \in \{1, 2, \ldots n\},$$
$$p_{4,0} = 1, \quad p_{4,1} = -5, \quad p_{4,k} = 0 \quad \forall\, k \in \{2, 3, \ldots n\},$$

$$q_{4,0} = \frac{9}{2}, \quad q_{4,1} = -\frac{1}{2}, \quad q_{4,k} = 0 \quad \forall\, k \in \{2, 3, \ldots n\}.$$

The values of $p_{j,k}$ and $q_{j,k}$ for $j \in \{1, 2, 3, 4\}$ and $k \in \{0, 1, \ldots n\}$ shown above are the same as those given by the formulae in (64) and (65). We will now show via mathematical induction that the values of $p_{j,k}$ and $q_{j,k}$ for $j \in \{5, 6, \ldots n\}$ and $k \in \{0, 1, \ldots n\}$ are also given by the formulae in (64) and (65).

Suppose that for some $4 \leq \bar{m} < n$ the values of $p_{j,k}$ and $q_{j,k}$ for all $j \in \{4, 5, \ldots \bar{m}\}$ and $k \in \{0, 1, \ldots n\}$ are given by the formulae in (64) and (65). Taking $j = \bar{m} - 1$ in (51) we get

$$[v_n]_{\bar{m}+1} = -h^4[\ddot{v}_n]_{\bar{m}-1} - [v_n]_{\bar{m}-3} + 4[v_n]_{\bar{m}-2} - 6[v_n]_{\bar{m}-1} + 4[v_n]_{\bar{m}}. \tag{A.6}$$

Substituting the parameterizations for $[v_n]_{\bar{m}-3}$, $[v_n]_{\bar{m}-2}$, $[v_n]_{\bar{m}-1}$, $[v_n]_{\bar{m}}$ and $[v_n]_{\bar{m}+1}$ from (A.5) into the above equation and comparing the coefficients of $h^{4k}y_1^{(2k)}$ and $h^{4k+2}y_2^{(2k)}$ on both sides of the resulting equation we get

$$p_{\bar{m}+1,k} = -p_{\bar{m}-1,k-1} - p_{\bar{m}-3,k} + 4\, p_{\bar{m}-2,k} - 6\, p_{\bar{m}-1,k} + 4\, p_{\bar{m},k}, \tag{A.7}$$

$$q_{\bar{m}+1,k} = -q_{\bar{m}-1,k-1} - q_{\bar{m}-3,k} + 4\, q_{\bar{m}-2,k} - 6\, q_{\bar{m}-1,k} + 4\, q_{\bar{m},k} \tag{A.8}$$

for all $k \in \{0, 1, \ldots n\}$. Here $p_{\bar{m}-1,-1} = q_{\bar{m}-1,-1} = 0$. The induction assumption implies that the values of the coefficients $p_{\bar{m}-1,k-1}$, $p_{\bar{m}-3,k}$, $p_{\bar{m}-2,k}$, $p_{\bar{m}-1,k}$ and $p_{\bar{m},k}$ are given by the formula in (64). Replacing these coefficients on the right side of (A.7) using the formula from (64) and then simplifying the resulting summation using the identity (A.4) with $l = \bar{m} + 2k - 4$ and $m = 4k$ we get

$$p_{\bar{m}+1,k} = (-1)^k \binom{\bar{m} + 2k}{4k} = (-1)^k \binom{(\bar{m}+1) + 2k - 1}{4k}$$

for all $k \in \{0, 1, \ldots n\}$. Hence the value of $p_{\bar{m}+1,k}$ is also given by the formula in (64). So we have shown that $p_{4,k}$ is given by (64) and if $p_{j,k}$ for all $j \in \{4, 5, \ldots \bar{m}\}$ and some $\bar{m} < n$ is given by (64), then $p_{\bar{m}+1,k}$ is also given by (64). So it follows from the principle of mathematical induction that the value of $p_{j,k}$ for all $j \in \{4, 5, \ldots n\}$ and $k \in \{0, 1, \ldots n\}$ is given by the formula in (64). This completes the verification of the formula for $p_{j,k}$ in (64). The formula for $q_{j,k}$ in (65) can be verified analogously via mathematical induction using (A.8).

C. ADDITIONAL NOTATIONS

Recall the notation EI and h from Section III. The constants $\sigma_{i,j}$ appearing in (96)-(100) are defined as follows:

TABLE 1. The definitions of the constants $\sigma_{i,j}$ in (96)-(100).

Notation	Definition
$\sigma_{1,1}$	$-\left(2EI(0) + \frac{2h}{3}EI_x(0) + EI(h)\right)/EI(h)$
$\sigma_{1,2}$	$\left(2EI(0) + \frac{2h}{3}EI_x(0) + 2EI(h)\right)/EI(h)$
$\sigma_{2,1}$	$\left(2EI(0) + \frac{2h}{3}EI_x(0) + 2EI(h)\right)/EI(2h)$
$\sigma_{2,2}$	$-\left(2EI(0) + \frac{2h}{3}EI_x(0) + 4EI(h) + EI(2h)\right)/EI(2h)$
$\sigma_{j,0}$	$-1/EI(jh) \quad \forall\, j \in \{1, 2, \ldots n-2\}$
$\sigma_{j,j-2}$	$-EI(jh-2h)/EI(jh) \quad \forall\, j \in \{3, 4, \ldots n-2\}$
$\sigma_{j,j-1}$	$(2EI(jh-2h) + 2EI(jh-h))/EI(jh) \quad \forall\, j \in \{3, 4, \ldots n-2\}$
$\sigma_{j,j}$	$-\left(EI(jh-2h)+4EI(jh-h)+EI(jh)\right)/EI(jh) \quad \forall\, j \in \{3, 4, \ldots, n-2\}$
$\sigma_{j,j+1}$	$\left(2EI(jh-h) + 2EI(jh)\right)/EI(jh) \quad \forall\, j \in \{2, 3, \ldots, n-2\}$
$\sigma_{n-1,n-3}$	$-EI(1-2h)$
$\sigma_{n-1,n-2}$	$2EI(1-2h) + 2EI(1-h)$
$\sigma_{n-1,n-1}$	$-EI(1-2h) - 4EI(1-h) - 2EI(1) + \frac{2h}{3}EI_x(1)$
$\sigma_{n-1,n}$	$2EI(1-h) + 2EI(1) - \frac{2h}{3}EI_x(1)$
$\sigma_{n,n-2}$	$-EI(1-h)$
$\sigma_{n,n-1}$	$2EI(1-h) + 2EI(1) - \frac{2h}{3}EI_x(1)$
$\sigma_{n,n}$	$-EI(1-h) - 2EI(1) + \frac{2h}{3}EI_x(1)$

D. PROOF OF $a_{n,0} = 0$, $c_{n,0} = 0$

From the procedure described above (60) for deriving (60)-(61), it is evident that

$$a_{n,0} = \frac{1}{h^3}(p_{n,0} - 3\, p_{n-1,0} + 3\, p_{n-2,0} - p_{n-3,0}),$$
$$c_{n,0} = \frac{1}{6h}(11p_{n,0} - 18p_{n-1,0} + 9\, p_{n-2,0} - 2p_{n-3,0})$$

for $n \geq 5$. Letting $k = 0$ in (64) we get $p_{j,0} = 1$. Using this in the above expressions we get $a_{n,0} = c_{n,0} = 0$.

E. PROOF OF $b_{n,0} = 0$, $d_{n,0} = 1$

From the procedure described above (60) for deriving (60)-(61), it is evident that

$$b_{n,0} = \frac{1}{h}(q_{n,0} - 3\, q_{n-1,0} + 3\, q_{n-2,0} - q_{n-3,0}), \tag{A.9}$$
$$d_{n,0} = \frac{h}{6}(11q_{n,0} - 18q_{n-1,0} + 9\, q_{n-2,0} - 2q_{n-3,0}) \tag{A.10}$$

for $n \geq 5$. Letting $k = 0$ in (65) we get

$$q_{j,0} = \frac{(j-1)^2}{2}.$$

Using this in (A.9) and (A.10) we get $b_{n,0} = 0$ and $d_{n,0} = (n-1)h$. Since $(n-1)h = 1$, it follows that $d_{n,0} = 1$.

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